Journal of Applied Analysis and Computation
Volume 17, Number 3, June 2027, 1792–1812
Website:http://www.jaac-online.com
DOI:10.11948/20250319

A STUDY OF COUPLED FRACTIONAL INTEGRODIFFERENTIAL SYSTEMS IN THE TEMPERED (K,φ)-HILFER FRAMEWORK WITHIN BANACH SPACES

Abdelkader Moumen[1,†], Oualid Zentar[2], Mahrouz Tayeb[2], Mohamed Ziane[2], Hicham Saber[1], Tariq Alraqad[1] and Etaf Saleh Alshawarbeh[1]

Abstract This work investigates a class of nonlinear coupled fractional integrodifferential systems formulated in the tempered (k,φ)-Hilfer framework. The analysis is conducted within an infinite-dimensional Banach space, where the nonlinear components are assumed to satisfy a blend of compactness and growth conditions. Employing the Mönch fixed-point theorem in conjunction with the measure of noncompactness, we establish an existence result that significantly generalizes and strengthens several existing results in the literature. To illustrate the applicability of the theoretical results, a representative example is provided in an infinite-dimensional Banach space setting.

Keywords Tempered fractional calculus, integro-differential equations, boundary value problems, Mönch fixed point theorem, measure of noncompactness.

MSC(2010) 34A08, 34B15, 47H08.

1. Introduction

The theory of fractional differential equations has been broadly applied in numerous theoretical and practical problems [6, 14, 19–21, 30]. In the literature one can find many definitions of fractional derivative and integral operators, each with its own worth, including those introduced by Riemann-Liouville, Hadamard, Riesz, Erdelyi-Kober and Caputo [8]. In [9], the authors introduced the (k,φ)-Hilfer derivative, based on the definition of the fractional k-Riemann-Liouville operator [15], and thus unifying and generalizing existing notions of fractional integral operators.

Meanwhile, by multiplying the kernel of the k-Riemann-Liouville fractional operator with an exponential factor, the expression known as the tempered (k,φ)-fractional derivative is naturally obtained. This extended derivative intervene in several fields, including geophysics [12] and finance [4] among others. Using different methods, various studies have been conducted on

†The corresponding author.
[1]Department of Mathematics, College of Science, University of Ha'il, Ha'il 55473, Saudi Arabia
[2]Department of Mathematics, University of Tiaret, Tiaret, Algeria and Laboratory of Research in Artificial Intelligence and Systems (LRAIS), University of Tiaret, Algeria
Email: mo.abdelkader@uoh.edu.sa(A. Moumen), oualid.zentar@univ-tiaret.dz(O. Zentar), tayeb.mahrouz@univ-tiaret.dz(M. Tayeb), mohamed.ziane@univ-tiaret.dz(M. Ziane), hi.saber@uoh.edu.sa(H. Saber), t.alraqad@uoh.edu.sa(T. Alraqad), e.alshawarbeh@uoh.edu.sa(E. S. Alshawarbeh)

the qualitative and quantitative aspects of initial/boundary value problems of fractional type determined by the tempered fractional derivative [17,28,32].

Additionally, coupled systems of fractional-order type arise naturally as the continuous counterpart of hybrid and switched dynamics, in which several fractional subsystems interact through their nonlinearities [11]. Such couplings provide a setting for the modelling of behaviours of hybrid nature combining memory effects and mode interaction, and they are therefore well suited to applications involving frequent mode transitions, as encountered in robotic control architectures, power systems and hybrid electric vehicles [23]. Coupled fractional systems of this kind, possibly with impulses or switching laws superimposed, have been studied in [26,27]; the present work is concerned with the underlying coupled model itself, no switching signal or switching law being assumed.

In [25], Thabet et al. obtained results on the existence of solutions to fractional integrodifferential equations for the class of Caputo FDEs:

$$\begin{cases} \mathcal{D}_{a^+}^{\vartheta_1}\mathfrak{u}_1(\sigma) = g_1\big(\sigma, \mathfrak{u}(\sigma), (\overline{\mathcal{H}}\mathfrak{u}_1)(\sigma)\big), & \sigma \in (0,b], \\ \mu_1\mathfrak{u}_1(a^+) + \nu_1\mathfrak{u}_1(b) = \eta_1, \end{cases}$$

where $\mathcal{D}_{a^+}^{\vartheta_1}$ is the Caputo fractional derivative of order $0 < \vartheta_1 \le 1$, $g : [0,b] \times \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ is a given function and μ_1, ν_1, η_1 are real numbers with $\mu_1 + \nu_1 \neq 0$, and $\mathcal{H}$ is a nonlinear integral operator given by $(\overline{\mathcal{H}}\mathfrak{u}_1)(\sigma) = \int_a^\sigma L(\sigma, s, \mathfrak{u}_1(s))ds$, where $L \in C(\mathcal{L} \times \mathcal{L} \times \mathbb{K}, \mathbb{K})$.

Thabet et al. in [24] investigated the existence of a solution to boundary value problems (for short, BVPs) for Hilfer FDEs by using the Mönch fixed point theorem:

$$\begin{cases} \mathcal{D}_{a^+}^{\vartheta_1,\kappa_1}\mathfrak{u}_1(\sigma) = g_1\big(\sigma, \mathfrak{u}(\sigma), (\widehat{\mathcal{H}}\mathfrak{u}_1)(\sigma)\big), & \sigma \in (a,b], \\ \mu_1\left(\mathcal{I}_{a^+}^{1-\gamma_1}\mathfrak{u}_1\right)(a^+) + \nu_1\left(\mathcal{I}_{a^+}^{1-\gamma_1}\mathfrak{u}_1\right)(b) = \eta_1, \end{cases}$$

where $\mathcal{D}_{a^+}^{\vartheta_1,\kappa_1}$ is the left-sided Hilfer fractional derivative of order $0 < \vartheta_1 < 1$ and type $0 \le \kappa_1 \le 1$, $\mu_1, \nu_1, \eta_1 \in \mathbb{R}$, where $\mu_1 + \nu_1 \neq 0$, and $g : [a,b] \times \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, $\mathbb{K}$ is an abstract Banach space, and $\mathcal{H}$ is a linear integral operator given by $(\widehat{\mathcal{H}}\mathfrak{u}_1)(\sigma) = \int_a^\sigma \widehat{L}(\sigma, s)\mathfrak{u}_1(s)ds$, with $\widehat{L} \in C(\mathcal{L} \times \mathcal{L}, \mathbb{R})$.

Let $(\mathbb{K}, \|\cdot\|)$ be Banach space. In this article the following coupled fractional integrodifferential system of tempered (k,φ)-Hilfer type is treated:

$$\begin{cases} {}_k^T\mathcal{D}_{a^+}^{\vartheta_1,\kappa_1,\varrho;\varphi}\mathfrak{u}_1(\sigma) = g_1(\sigma, \mathfrak{u}_1(\sigma), (\mathcal{H}\mathfrak{u}_2)(\sigma)), \quad \sigma \in \mathcal{L}' := (a,b], \\ {}_k^T\mathcal{D}_{a^+}^{\vartheta_2,\kappa_2,\varrho;\varphi}\mathfrak{u}_2(\sigma) = g_2(\sigma, \mathfrak{u}_2(\sigma), (\mathcal{H}\mathfrak{u}_1)(\sigma)), \quad \sigma \in \mathcal{L}' := (a,b], \\ \mu_1\left({}_k^T\mathcal{I}_{a^+}^{k(1-\gamma_1),\varrho;\varphi}\mathfrak{u}_1\right)(a^+) + \nu_1\left({}_k^T\mathcal{I}_{a^+}^{k(1-\gamma_1),\varrho;\varphi}\mathfrak{u}_1\right)(b) = \eta_1, \\ \mu_2\left({}_k^T\mathcal{I}_{a^+}^{k(1-\gamma_2),\varrho;\varphi}\mathfrak{u}_2\right)(a^+) + \nu_2\left({}_k^T\mathcal{I}_{a^+}^{k(1-\gamma_2),\varrho;\varphi}\mathfrak{u}_2\right)(b) = \eta_2, \end{cases} \tag{1.1}$$

where for $k > 0$ we denote by ${}_k^T\mathcal{D}_{a^+}^{\vartheta_i,\kappa_i,\varrho;\varphi}$, $i = 1,2$, the tempered (k,φ)-Hilfer FD with order $0 < \vartheta_i < k$, having type $\kappa_i \in [0,1]$ and index $\varrho \in [0,\infty)$, while ${}_k^T\mathcal{I}_{a^+}^{k(1-\gamma_i),\varrho;\varphi}$ stands for the tempered fractional (k,φ)-R-L integral of order $k(1-\gamma_i) > 0$, $\gamma_i = \frac{1}{k}(\vartheta_i + \kappa_i(k-\vartheta_i))$ and $k(1-\gamma_j) < \vartheta_i$ for $i,j = 1,2$. Throughout the paper, γ_i always denotes this quantity; it is the critical index governing the behaviour of the solutions at the left endpoint a^+ and it is precisely the index used in the definition of the weighted spaces $C_{1-\gamma_i}$ introduced in Section

2. $\mu_i, \nu_i, \eta_i \in \mathbb{R}$, where $\mu_i + \nu_i e^{-\varrho(\varphi(b)-\varphi(a))} \neq 0$, and $g_i : [a,b] \times \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ are suitably chosen functions, the precise specifications of which will be provided in subsequent sections, and $\mathcal{H}$ is a nonlinear integral operator given by $(\mathcal{H}\mathfrak{u}_i)(\sigma) = \int_a^\sigma L(\sigma, s, \mathfrak{u}_i(s))\varphi'(s)ds$, where $L \in C(\mathcal{L} \times \mathcal{L} \times \mathbb{K}, \mathbb{K})$.

This paper is structured as follows: In Section 2, we give of some fundamental preliminaries needed in the sequel. In Section 3, Mönch's fixed-point principle is used to establish a new existence result under a regularity-type condition expressed in terms of the Hausdorff measure of noncompactness. Finally, an illustrative example in the infinite dimensional Banach space setting is also provided in Section 4.

2. Preliminaries

In what follows, let $\mathcal{L} := [a,b]$, we equip the space of $\mathbb{K}$-valued continuous functions, $C(\mathcal{L}, \mathbb{K})$, on the interval $\mathcal{L}$ with the sup norm

$$\|\mathfrak{u}\|_\infty = \sup_{\sigma \in \mathcal{L}} |\mathfrak{u}(\sigma)|. \tag{2.1}$$

For ease of notation, we define

$$\mathbb{S}^1_+(\mathcal{L}, \mathbb{R}) = \{\varphi \ : \ \varphi \in C^1(\mathcal{L}, \mathbb{R}) \ \text{ and } \ \varphi'(\sigma) > 0 \ \text{ for all } \ \sigma \in \mathcal{L}\}.$$

For $\varphi \in \mathbb{S}^1_+(\mathcal{L}, \mathbb{R})$ and $\sigma, s \in \mathcal{L}$, $(\sigma > s)$, we set

$$\omega(\sigma, s) = \varphi(\sigma) - \varphi(s) \ \text{ and } \ \omega(\sigma, s)^\alpha = (\varphi(\sigma) - \varphi(s))^\alpha.$$

Define the weighted space

$$C_{1-\gamma_i}(\mathcal{L}, \mathbb{K}) = \{\mathfrak{u} \in C(\mathcal{L}', \mathbb{K}) : \omega(\cdot, a)^{1-\gamma_i}\mathfrak{u}(\cdot) \in C(\mathcal{L}, \mathbb{K})\},$$

endowed with

$$\|\mathfrak{u}\|_{\gamma_i} = \sup_{\sigma \in \mathcal{L}} \omega(\sigma, a)^{1-\gamma_i}\|\mathfrak{u}(\sigma)\|.$$

Henceforth, for $\mho \subset C_{1-\gamma_i}(\mathcal{L}, \mathbb{K})$, we represent the set $\mho_{\gamma_i}$ by

$$\mho_{\gamma_i} = \{\mathfrak{u}_{\gamma_i} \ : \ \mathfrak{u} \in \mho\},$$

where

$$\mathfrak{u}_{\gamma_i}(\sigma) = \begin{cases} \omega(\sigma, a)^{1-\gamma_i}\mathfrak{u}(\sigma), & \sigma \in \mathcal{L}', \\ \lim_{\sigma \to a^+} \omega(\sigma, a)^{1-\gamma_i}\mathfrak{u}(\sigma), & \sigma = a. \end{cases} \tag{2.2}$$

Also, the weighted space of Bochner integrable functions $\mathfrak{u} : \mathcal{L} \to \mathbb{K}$, endowed with the norm

$$\|\mathfrak{u}\|_{L^p_\varphi} = \left(\int_a^b \varphi'(s)\|\mathfrak{u}\|^p ds\right)^{\frac{1}{p}} \tag{2.3}$$

will be denoted by $L^p_\varphi(\mathcal{L}, \mathbb{K})$ $(1 \leq p < \infty)$. In the case $\varphi(\sigma) = \sigma$, this space is simply equal to $L^p(\mathcal{L}, \mathbb{Y})$.

Definition 2.1. [5] For $\alpha, \theta, k > 0$, the k-beta and k-gamma functions are defined by

$$B_k(\alpha,\theta) = \frac{1}{k}\int_0^1 \sigma^{\frac{\alpha}{k}-1}(1-\sigma)^{\frac{\theta}{k}-1}d\sigma,$$
$$\Gamma_k(\alpha) = \int_0^\infty \sigma^{\alpha-1}e^{-\frac{\sigma^k}{k}}d\sigma.$$

As a consequence, we have

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \quad \Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha), \quad \Gamma_k(k) = \Gamma(1) = 1.$$

So that

$$B_k(\alpha,\theta) = \frac{1}{k}B\left(\frac{\alpha}{k},\frac{\theta}{k}\right) \quad \text{and} \quad B_k(\alpha,\theta) = \frac{\Gamma_k(\alpha)\Gamma_k(\theta)}{\Gamma_k(\alpha+\theta)},$$

where $B(\cdot,\cdot)$ denotes the classical Euler beta function. Throughout the paper the k-beta function is systematically written with the subscript k, i.e. $B_k(\cdot,\cdot)$, so as to avoid any confusion with $B(\cdot,\cdot)$.

Definition 2.2. [22] Let $\varphi \in \mathbb{S}^1_+(\mathcal{L},\mathbb{R})$, $\alpha, \varrho \in [0,\infty)$, and $k > 0$. The tempered (k,φ)-RLFI of a function $\mathfrak{u} \in C(\mathcal{L},\mathbb{R})$ is defined by

$${}^T_k\mathcal{I}^{\alpha,\varrho;\varphi}_{a^+}\mathfrak{u}(\sigma) = e^{-\varrho\omega(\sigma,a)}\,{}_k\mathcal{I}^{\alpha;\varphi}_{a^+}\left(\mathfrak{u}(\sigma)e^{\varrho\omega(\sigma,a)}\right) = \frac{1}{k\Gamma_k(\alpha)}\int_a^\sigma \varphi'(s)\mathcal{K}^{\frac{\alpha}{k}-1}_{\varrho,\omega}(\sigma,s)\mathfrak{u}(s)ds,$$

where $\mathcal{K}^{\frac{\alpha}{k}-1}_{\varrho,\omega}(\sigma,s) = e^{-\varrho\omega(\sigma,s)}\omega(\sigma,s)^{\frac{\alpha}{k}-1}$ and ${}_k\mathcal{I}^{\alpha;\varphi}_{a^+}$ is the (k,φ)-fractional integral [18], defined by

$${}_k\mathcal{I}^{\alpha;\varphi}_{a^+}\mathfrak{u}(\sigma) = \frac{1}{k\Gamma_k(\alpha)}\int_a^\sigma \varphi'(s)\omega(\sigma,s)^{\frac{\alpha}{k}-1}\mathfrak{u}(s)ds.$$

Lemma 2.1. [22] *Let $\alpha_1, \alpha_2, k > 0$ and $\varrho \in [0,\infty)$, then we have*

1. ${}^T_k\mathcal{I}^{\alpha_1,\varrho;\varphi}_{a^+}\,{}^T_k\mathcal{I}^{\alpha_2,\varrho;\varphi}_{a^+}\mathfrak{u}(\sigma) = {}^T_k\mathcal{I}^{\alpha_2,\varrho;\varphi}_{a^+}\,{}^T_k\mathcal{I}^{\alpha_1,\varrho;\varphi}_{a^+}\mathfrak{u}(\sigma) = {}^T_k\mathcal{I}^{\alpha_1+\alpha_2,\varrho;\varphi}_{a^+}\mathfrak{u}(\sigma)$.
2. ${}^T_k\mathcal{I}^{\alpha_1,\varrho;\varphi}_{a^+}\mathcal{K}^{\frac{\alpha_2}{k}-1}_{\varrho,\omega}(\sigma,a) = \frac{\Gamma_k(\alpha_2)}{\Gamma_k(\alpha_1+\alpha_2)}\mathcal{K}^{\frac{\alpha_1+\alpha_2}{k}-1}_{\varrho,\omega}(\sigma,a)$.

Definition 2.3. [22] Let $\varphi \in \mathbb{S}^1_+(\mathcal{L},\mathbb{R})$ and $k > 0$. The (k,φ)-Hilfer tempered FD of $\mathfrak{u} \in C(\mathcal{L},\mathbb{R})$ of order $0 < \frac{\alpha}{k} < 1$, type $0 \le \beta \le 1$ and index $\varrho \in [0,\infty)$ is defined by

$${}^T_k\mathcal{D}^{\alpha,\beta,\varrho;\varphi}_{a^+}\mathfrak{u}(\sigma) = \left({}^T_k\mathcal{I}^{\beta(k-\alpha),\varrho;\varphi}_{a+}\left(\frac{1}{\varphi'(\sigma)}\frac{d}{d\sigma} + \varrho\right)\left(k\,{}^T_k\mathcal{I}^{(1-\beta)(k-\alpha),\varrho;\varphi}_{a+}\mathfrak{u}\right)\right)(\sigma).$$

Lemma 2.2. [31, Lemma 1] *A set $\mho \subset C_{1-\gamma_i}(\mathcal{L},\mathbb{K})$ is relatively compact if and only if $\mho_{\gamma_i}$ is relatively compact in $C(\mathcal{L},\mathbb{K})$.*

Next, we denote the closed convex hull of A by $\overline{\text{conv}}\,A$, and $\mathcal{P}(\mathbb{K})$ will stand for the family of all bounded subsets in $\mathbb{K}$.

Definition 2.4. [2] The Hausdorff MNC is the map $\Upsilon : \mathcal{P}(\mathbb{K}) \longrightarrow [0,\infty)$ defined by:

$$\Upsilon(A) = \inf\{\epsilon > 0 : A \text{ has a finite } \epsilon - \text{net in } \mathbb{K}\}.$$

The subsequent properties are established; for further elaboration, refer to [2, 10].

1) $\Upsilon(A) = 0$ if and only if $\overline{A}$ is compact.
2) $\Upsilon(A) = \Upsilon(\overline{A}) = \Upsilon(\overline{\text{conv}}A)$;
3) $\Upsilon(dA) = |d|\Upsilon(A)$ for every $d \in \mathbb{R}$;
4) $\Upsilon(A) \leq \Upsilon(B)$ if $A \subset B$;
5) $\Upsilon(A + B) \leq \Upsilon(A) + \Upsilon(B)$;
6) If $T : \mathbb{Y} \longrightarrow \mathbb{Y}$ is a bounded linear operator, then $\Upsilon(TA) \leq \|T\|_{\mathcal{B}(\mathbb{Y})}\Upsilon(A)$;
7) If $\{A_n\}_{n\geq 1}$ is a decreasing sequence of bounded closed nonempty subsets of $\mathbb{Y}$ and $\lim_{n\to\infty} \Upsilon(A_n) = 0$ then $\bigcap_{n=1}^{\infty} A_n$ is a nonempty and compact subset of $\mathbb{Y}$.

Lemma 2.3. [29] *Let $\mho \subseteq C_{1-\gamma_i}(\mathcal{L}, \mathbb{K})$. Then, for all $\sigma \in \mathcal{L}$, one has*

$$\Upsilon(\mho_{\gamma_i}(\sigma)) \leq \Upsilon_{C_{1-\gamma_i}}(\mho).$$

Moreover, if $\mho$ is equicontinuous on $\mathcal{L}$, one obtains that $\Upsilon(\mho_{\gamma_i}(\cdot))$ is continuous on I and

$$\Upsilon_{C_{1-\gamma_i}}(\mho) = \sup_{\sigma\in\mathcal{L}} \Upsilon(\mho_{\gamma_i}(\sigma)).$$

Lemma 2.4. [3, 7] *Let $\{u_n\}_{n=1}^{+\infty} \subset L^1(\mathcal{L}, \mathbb{K})$ satisfy $\|u_n(\sigma)\| \leq \iota(\sigma)$ a.e. on $\mathcal{L}$ for all $n \geq 1$ with some $\iota \in L^1(\mathcal{L}, \mathbb{R}_+)$. Then the function $\Upsilon(\{u_n(\sigma)\}_{n=1}^{+\infty})$ belongs to $L^1(\mathcal{L}, \mathbb{R}_+)$ and*

$$\Upsilon\left(\left\{\int_a^{\sigma} u_n(s)ds \ : \ n \geq 1\right\}\right) \leq 2\int_a^{\sigma} \Upsilon\left(\{u_n(s) \ : \ n \geq 1\}\right) ds. \tag{2.4}$$

Theorem 2.1. [13] *Let $\mathbb{P}$ be a closed convex subset of a Banach space $\mathbb{K}$ with $0 \in \mathbb{P}$. Suppose that $\mathcal{T} : \mathbb{P} \to \mathbb{K}$ is a continuous map satisfying the Mönch's condition*

$$\left(\begin{array}{c} \text{if } \mathbb{J} \subseteq \mathbb{P} \text{ countable, and} \\ \mathbb{J} \subseteq \text{conv}(\mathcal{T}(\mathbb{J}) \cup \{0\}) \end{array}\right) \Longrightarrow \overline{\mathbb{J}} \text{ is compact.} \tag{2.5}$$

Then, $\mathcal{T}$ has a fixed point in $\mathbb{P}$.

3. Main results

The principal result established herein is the existence of a solution to problem (1.1). To proceed, we introduce the following hypotheses.

(H1) The functions $g_i : \mathcal{L} \times \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ satisfy the Carathéodory-type conditions; i.e., (i) for all $x, y \in \mathbb{K}$ $g_i(\cdot, x, y) : \mathcal{L} \to \mathbb{K}$ is a measurable function and (ii) for a.e. $\sigma \in \mathcal{L}$ $g_i(\sigma, \cdot, \cdot) : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ is a continuous function.

(H2) The function $L : \mathcal{L} \times \mathcal{L} \times \mathbb{K} \to \mathbb{K}$ satisfies the Carathéodory-type conditions; i.e., for all $x \in \mathbb{K}$ $L(\cdot, \cdot, x)$ is measurable and for a.e. $s \in [a, \sigma]$, $\sigma \in \mathcal{L}$ the function $L(\sigma, s, \cdot)$ is continuous.

(H3) There exist constants $M_i, M_L > 0$, such that

$$\|g_i(\sigma, \mathfrak{u}_i(\sigma), (\mathcal{H}\mathfrak{u}_{3-i})(\sigma))\| \leq M_i \left(1 + \|\mathfrak{u}_i(\sigma)\| + \|(\mathcal{H}\mathfrak{u}_{3-i})(\sigma)\|\right), \quad i = 1, 2,$$

and

$$\|L(\sigma, s, \mathfrak{u}_i(\sigma))\| \leq M_L(1 + \|\mathfrak{u}_i(\sigma)\|), \quad i = 1, 2,$$

for all $\sigma \in \mathcal{L}$ and each $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathbb{K}$.

(H4) There exist real-valued functions $\Psi_{i,j}, \sigma_i \in L^\infty(\mathcal{L}, \mathbb{R})$ such that

$$\Upsilon\left(g_i(\sigma, D^1, D^2)\right) \leq \sum_{j=1}^{2} \Psi_{i,j}(\sigma)\Upsilon(D^i), \quad i = 1, 2,$$

and

$$\Upsilon\left(L(\sigma, s, D^i)\right) \leq \sigma_i(\sigma)\Upsilon(D^i), \quad i = 1, 2,$$

for bounded sets $D^1, D^2 \subset \mathbb{K}$, a.e. $\sigma \in \mathcal{L}$.

(H5) Suppose that for $i = 1, 2$

$$1 \geq \frac{M_i e^{\varrho\omega(b,a)}}{\Gamma_k(\vartheta_i)} \Bigg(\omega(b,a)^{\frac{\vartheta_i}{k}} B_k\left(\vartheta_i, k\gamma_i\right) + \frac{M_L}{\gamma_{3-i}} \omega(b,a)^{\frac{\vartheta_i + k + k(\gamma_{3-i} - \gamma_i)}{k}} B_k\left(\vartheta_i, k\gamma_{3-i} + k\right)$$
$$+ \frac{|\nu_i|\omega(b,a)^{\frac{\vartheta_i + k\gamma_i}{k} - 1}}{|\Pi_i(b, \gamma_i)|} B_k\left(\vartheta_i, k\gamma_i\right) + \frac{M_L|\nu_i|\omega(b,a)^{\frac{\vartheta_i + k\gamma_{3-i}}{k}}}{\gamma_{3-i}|\Pi_i(b, \gamma_i)|} B_k\left(\vartheta_i, k\gamma_{3-i} + k\right) \Bigg),$$

where $\Pi_i(b, \gamma_i) := \Gamma_k(k\gamma_i)\left(\mu_i + \nu_i e^{-\varrho\omega(b,a)}\right)$, $i = 1, 2$.

To simplify the ensuing computations, we define

$$\Psi^*_{i,j} = \|\Psi_{i,j}\|_{L^\infty}, \quad \sigma^*_{3-i} = \|\sigma_{3-i}\|_{L^\infty}, \; i, j = 1, 2.$$

Remark 3.1. The right-hand side of (H5) is a finite positive constant which depends only on the endpoints a, b (through $\omega(b,a) = \varphi(b) - \varphi(a)$), on the fractional parameters $\vartheta_i, \kappa_i, k, \varrho$ (through γ_i, Γ_k and the k-beta functions $B_k(\cdot,\cdot)$) and on the growth constants M_i, M_L. In particular, all the k-beta functions occurring in (H5) are evaluated at strictly positive arguments, since $\vartheta_i > 0$, $k\gamma_i > 0$ and $k\gamma_{3-i} + k > 0$; hence they are finite, and so is the whole expression.

Theorem 3.1. *Assume that (H1)-(H5) are satisfied. Then the system (1.1) possesses at least one solution on $\mathcal{L}$.*

Proof. First, we endow the product space $C_{\gamma_1,\gamma_2}(\mathcal{L}, \mathbb{K}) := C_{\gamma_1}(\mathcal{L}, \mathbb{K}) \times C_{\gamma_2}(\mathcal{L}, \mathbb{K})$ with the norm

$$\|(\mathfrak{u}_1, \mathfrak{u}_2)\|_{\gamma_1,\gamma_2} = \|\mathfrak{u}_1\|_{\gamma_1} + \|\mathfrak{u}_2\|_{\gamma_2}. \tag{3.1}$$

In order to avoid any conflict of notation with the interval $\mathcal{L} = [a, b]$ and with the abstract map appearing in Theorem 2.1, the solution operator is denoted by $\mathcal{A}$ throughout this section. Based on [22, Theorem 4.1], we see the equivalence between the system (1.1) and the operator equation $\mathcal{A}(\mathfrak{z}_1, \mathfrak{z}_2) = (\mathfrak{z}_1, \mathfrak{z}_2)$ where $\mathcal{A} : C_{\gamma_1,\gamma_2}(\mathcal{L}, \mathbb{K}) \to C_{\gamma_1,\gamma_2}(\mathcal{L}, \mathbb{K})$ is the operator given by:

$$\mathcal{A}(\mathfrak{u}_1(\sigma), \mathfrak{u}_2(\sigma)) = (\mathcal{A}_1(\mathfrak{u}_1(\sigma), \mathfrak{u}_2(\sigma)), \mathcal{A}_2(\mathfrak{u}_1(\sigma), \mathfrak{u}_2(\sigma))) \tag{3.2}$$

where, for $i = 1, 2$,

$$\begin{aligned}
&\mathcal{A}_i(\mathfrak{u}_1(\sigma), \mathfrak{u}_2(\sigma)) \\
&= \frac{1}{k\Gamma_k(\vartheta_i)} \int_a^\sigma \varphi'(s) \mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma, s) g_i(s, \mathfrak{u}_i(s), (\mathcal{H}\mathfrak{u}_{3-i})(s)) ds \\
&+ \frac{\mathcal{K}_{\varrho,\omega}^{\gamma_i-1}(\sigma, a)}{\Pi_i(b, \gamma_i)} \left[\eta_i - \frac{\nu_i}{k\Gamma_k(\vartheta_i)} \int_a^b \varphi'(s) \mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-\gamma_i}(b, s) g_i(s, \mathfrak{u}_i(s), (\mathcal{H}\mathfrak{u}_{3-i})(s)) ds \right].
\end{aligned} \tag{3.3}$$

Define the closed ball

$$\mathbb{B}_R = \{(\mathfrak{u}_1, \mathfrak{u}_2) \in C_{\gamma_1,\gamma_2}(\mathcal{L}, \mathbb{K}) \ : \ \|\mathfrak{u}_1\|_{\gamma_1} \leq R, \ \|\mathfrak{u}_2\|_{\gamma_2} \leq R\}. \tag{3.4}$$

The argument proceeds via a stepwise construction, outlined as follows.

Step 1. The continuity of the operator $\mathcal{A}$.

Take a sequence $\{\mathfrak{u}_{1,n}, \mathfrak{u}_{2,n}\}$ satisfying $\{\mathfrak{u}_{1,n}, \mathfrak{u}_{2,n}\} \to (\mathfrak{u}_1, \mathfrak{u}_2)$ in $\mathbb{B}_R$ as $n \to \infty$. For each $\sigma \in \mathcal{L}$, by applying assumption (H1), it follows immediately that

$$\|g_i(s, \mathfrak{u}_{i,n}(s), (\mathcal{H}\mathfrak{u}_{3-i,n})(s)) - g_i(s, \mathfrak{u}_i(s), (\mathcal{H}\mathfrak{u}_{3-i})(s))\| \to 0, \quad \text{as} \ \ n \longrightarrow \infty, \quad i = 1, 2.$$

Next, in view of (H3), one gets

$$\begin{aligned}
\|(\mathcal{H}\mathfrak{u}_{3-i})(\sigma)\| &\leq \int_a^\sigma \|L(\sigma, s, \mathfrak{u}_{3-i}(s))\| \varphi'(s) ds \\
&\leq \int_a^\sigma M_L(1 + \|\mathfrak{u}_{3-i}(s)\|) \varphi'(s) ds \\
&\leq M_L \int_a^\sigma \varphi'(s) ds + M_L \|\mathfrak{u}_{3-i}\|_{1-\gamma_{3-i}} \int_a^\sigma \omega(s, a)^{\gamma_{3-i}-1} \varphi'(s) ds \\
&\leq M_L \left(\omega(\sigma, a) + \frac{\omega(\sigma, a)^{\gamma_{3-i}}}{\gamma_{3-i}} \|\mathfrak{u}_{3-i}\|_{1-\gamma_{3-i}} \right),
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\|g_i(s, \mathfrak{u}_{i,n}(s), (\mathcal{H}\mathfrak{u}_{3-i,n})(s)) - g_i(s, \mathfrak{u}_i(s), (\mathcal{H}\mathfrak{u}_{3-i})(s))\| \\
&\leq \|g_i(s, \mathfrak{u}_{i,n}(s), (\mathcal{H}\mathfrak{u}_{3-i,n})(s))\| + \|g_i(s, \mathfrak{u}_i(s), (\mathcal{H}\mathfrak{u}_{3-i})(s))\| \\
&\leq 2M_i \left(1 + \|\mathfrak{u}_i(s)\| + \|(\mathcal{H}\mathfrak{u}_{3-i})(s)\|\right) \\
&\leq 2M_i \left(1 + R\omega(s, a)^{\gamma_i - 1} + M_L \left(\omega(s, a) + R \frac{\omega(s, a)^{\gamma_{3-i}}}{\gamma_{3-i}} \right) \right), \quad i = 1, 2.
\end{aligned} \tag{3.6}$$

Since the function $s \mapsto \varphi'(s)\omega(s, a)^{\gamma_i - 1}$ is Lebesgue integrable over $[a, \sigma]$, so are the functions $s \mapsto \mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma, s)\varphi'(s)\omega(s, a)^{\gamma_i-1}$, and $s \mapsto \mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-\gamma_i}(b, s)\varphi'(s)\omega(s, a)^{\gamma_i-1}$, $i = 1, 2$. Then it follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned}
&\left\|\omega(\sigma, a)^{1-\gamma_i} \left(\mathcal{A}_i\big(\mathfrak{u}_{1,n}(\sigma), \mathfrak{u}_{2,n}(\sigma)\big) - \mathcal{A}_i\big(\mathfrak{u}_1(\sigma), \mathfrak{u}_2(\sigma)\big)\right)\right\| \\
&\leq \frac{\omega(\sigma, a)^{1-\gamma_i}}{k\Gamma_k(\vartheta_i)} \int_a^\sigma \varphi'(s) \mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma, s) \left\| g_i\big(s, \mathfrak{u}_{i,n}(s), (\mathcal{H}\mathfrak{u}_{3-i,n})(s)\big) - g_i\big(s, \mathfrak{u}_i(s), (\mathcal{H}\mathfrak{u}_{3-i})(s)\big) \right\| ds
\end{aligned}$$

$$+\int_a^b \frac{|\nu_i|\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-\gamma_i}(b,s)}{k\Gamma_k(\vartheta_i)\,|\Pi_i(b,\gamma_i)|}\Big\|g_i\big(s,\mathfrak{u}_{i,n}(s),(\mathcal{H}\mathfrak{u}_{3-i,n})(s)\big)-g_i\big(s,\mathfrak{u}_i(s),(\mathcal{H}\mathfrak{u}_{3-i})(s)\big)\Big\|\,ds$$
$$\longrightarrow 0,$$

as $n\to\infty$ for all $\sigma\in\mathcal{L}$. Therefore,

$$\|\mathcal{A}_i(\mathfrak{u}_{1,n}(\cdot),\mathfrak{u}_{2,n}(\cdot))-\mathcal{A}_i(\mathfrak{u}_1(\cdot),\mathfrak{u}_2(\cdot))\|_{1-\gamma_i}\xrightarrow[n\to\infty]{}0,\quad i=1,2.$$

Accordingly, we achieve the continuity of the operator $\mathcal{A}(\cdot,\cdot)$ as desired.

Step 2. The set $\mathbb{B}_R$ is mapped into itself by the operator $\mathcal{A}$.

Suppose that for each $R\in\mathbb{N}^*$ there is $(\mathfrak{u}_1,\mathfrak{u}_2)\in\mathbb{B}_R$ and $\sigma\in\mathcal{L}$ such that

$$\|\mathcal{A}_i(\mathfrak{u}_1,\mathfrak{u}_2)\|_{\gamma_i}>R,\quad i=1,2.$$

Now, for all $\sigma\in\mathcal{L}$, $(\mathfrak{u}_1,\mathfrak{u}_2)\in\mathbb{B}_R$. Using the fact that $e^{-\varrho\omega(\sigma,a)}\le 1$, for $\sigma\in\mathcal{L}$, and

$$e^{-\varrho\omega(\sigma,s)}=e^{-\varrho\omega(\sigma,a)}e^{\varrho\omega(s,a)}\le e^{\varrho\omega(b,a)}\text{ for }s\in[a,\sigma],\tag{3.7}$$

we have

$$\begin{aligned}&\omega(\sigma,a)^{1-\gamma_i}\left\|\mathcal{A}_i\big(\mathfrak{u}_1(\sigma),\mathfrak{u}_2(\sigma)\big)\right\|\\ =&\frac{e^{\varrho\omega(b,a)}\omega(\sigma,a)^{1-\gamma_i}}{k\Gamma_k(\vartheta_i)}\int_a^\sigma\varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\left\|g_i\big(s,\mathfrak{u}_i(s),(\mathcal{H}\mathfrak{u}_{3-i})(s)\big)\right\|\,ds+\frac{|\eta_i|}{|\Pi_i(b,\gamma_i)|}\\ &+\frac{|\nu_i|e^{\varrho\omega(b,a)}}{k\Gamma_k(\vartheta_i)|\Pi_i(b,\gamma_i)|}\int_a^b\varphi'(s)\omega(b,s)^{\frac{\vartheta_i}{k}-\gamma_i}\left\|g_i\big(s,\mathfrak{u}_i(s),(\mathcal{H}\mathfrak{u}_{3-i})(s)\big)\right\|\,ds,\qquad i=1,2.\end{aligned}$$

By the hypothesis (H3), we get

$$\begin{aligned}&\omega(\sigma,a)^{1-\gamma_i}\left\|\mathcal{A}_i\big(\mathfrak{u}_1(\sigma),\mathfrak{u}_2(\sigma)\big)\right\|\\ \le&\frac{e^{\varrho\omega(b,a)}\omega(\sigma,a)^{1-\gamma_i}}{k\Gamma_k(\vartheta_i)}\int_a^\sigma\varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}M_i\left(1+\|\mathfrak{u}_i(s)\|+\|(\mathcal{H}\mathfrak{u}_{3-i})(s)\|\right)ds+\frac{|\eta_i|}{|\Pi_i(b,\gamma_i)|}\\ &+\frac{|\nu_i|e^{\varrho\omega(b,a)}}{k\Gamma_k(\vartheta_i)|\Pi_i(b,\gamma_i)|}\int_a^b\varphi'(s)\omega(b,s)^{\frac{\vartheta_i}{k}-\gamma_i}M_i\left(1+\|\mathfrak{u}_i(s)\|+\|(\mathcal{H}\mathfrak{u}_{3-i})(s)\|\right)ds,\qquad i=1,2.\end{aligned}$$

By (3.5), we have

$$\begin{aligned}&\omega(\sigma,a)^{1-\gamma_i}\left\|\mathcal{A}_i\big(\mathfrak{u}_1(\sigma),\mathfrak{u}_2(\sigma)\big)\right\|\\ \le&\frac{M_ie^{\varrho\omega(b,a)}\omega(\sigma,a)^{1-\gamma_i}}{k\Gamma_k(\vartheta_i)}\int_a^\sigma\varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\bigg(1+R\omega(s,a)^{\gamma_i-1}+M_L\omega(s,a)\\ &+\frac{M_LR}{\gamma_{3-i}}\omega(s,a)^{\gamma_{3-i}}\bigg)ds+\frac{|\eta_i|}{|\Pi_i(b,\gamma_i)|}\\ &+\frac{M_i|\nu_i|e^{\varrho\omega(b,a)}}{k\Gamma_k(\vartheta_i)|\Pi_i(b,\gamma_i)|}\int_a^b\varphi'(s)\omega(b,s)^{\frac{\vartheta_i}{k}-\gamma_i}\bigg(1+R\omega(s,a)^{\gamma_i-1}+M_L\omega(s,a)\\ &+\frac{M_LR}{\gamma_{3-i}}\omega(s,a)^{\gamma_{3-i}}\bigg)ds,\qquad i=1,2.\end{aligned}\tag{3.8}$$

By a change of variables $\omega(s,a) = t\omega(\sigma,a)$ and using the definition of the k-beta function, we obtain, after simple calculations:

$$\begin{aligned}
&\int_a^\sigma \varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\left(1+R\omega(s,a)^{\gamma_i-1}+M_L\omega(s,a)+\frac{M_LR}{\gamma_{3-i}}\omega(s,a)^{\gamma_{3-i}}\right)ds\\
&=\int_a^\sigma \varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\,ds+R\int_a^\sigma \varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_i-1}\,ds\\
&\quad+M_L\int_a^\sigma \varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)\,ds+\frac{M_LR}{\gamma_{3-i}}\int_a^\sigma \varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_{3-i}}\,ds \qquad (3.9)\\
&=\frac{k}{\vartheta_i}\omega(\sigma,a)^{\frac{\vartheta_i}{k}}+kR\omega(\sigma,a)^{\frac{\vartheta_i+k\gamma_i}{k}-1}B_k(\vartheta_i,k\gamma_i)+kM_L\omega(\sigma,a)^{\frac{\vartheta_i+k}{k}}B_k(\vartheta_i,2k)\\
&\quad+k\frac{M_LR}{\gamma_{3-i}}\omega(\sigma,a)^{\frac{\vartheta_i+k\gamma_{3-i}}{k}}B_k(\vartheta_i,k\gamma_{3-i}+k),\qquad i=1,2.
\end{aligned}$$

Repeating this procedure again, we get

$$\begin{aligned}
&\int_a^b \varphi'(s)\omega(b,s)^{\frac{\vartheta_i}{k}-1}\left(1+R\omega(s,a)^{\gamma_i-1}+M_L\omega(s,a)+\frac{M_LR}{\gamma_{3-i}}\omega(s,a)^{\gamma_{3-i}}\right)ds\\
&=\frac{k}{\vartheta_i}\omega(b,a)^{\frac{\vartheta_i}{k}}+kR\omega(b,a)^{\frac{\vartheta_i+k\gamma_i}{k}-1}B_k(\vartheta_i,k\gamma_i)+kM_L\omega(b,a)^{\frac{\vartheta_i+k}{k}}B_k(\vartheta_i,2k) \qquad (3.10)\\
&\quad+k\frac{M_LR}{\gamma_{3-i}}\omega(b,a)^{\frac{\vartheta_i+k\gamma_{3-i}}{k}}B_k(\vartheta_i,k\gamma_{3-i}+k),\qquad i=1,2.
\end{aligned}$$

Substituting (3.9) and (3.10) into (3.8), we obtain

$$\begin{aligned}
&\omega(\sigma,a)^{1-\gamma_i}\left\|\mathcal{A}_i\big(\mathfrak{u}_1(\sigma),\mathfrak{u}_2(\sigma)\big)\right\|\\
&\le\frac{M_ie^{\varrho\omega(b,a)}\omega(\sigma,a)^{1-\gamma_i}}{k\Gamma_k(\vartheta_i)}\left[\frac{k}{\vartheta_i}\omega(\sigma,a)^{\frac{\vartheta_i}{k}}+kR\omega(\sigma,a)^{\frac{\vartheta_i+k\gamma_i}{k}-1}B_k(\vartheta_i,k\gamma_i)\right.\\
&\quad\left.+kM_L\omega(\sigma,a)^{\frac{\vartheta_i+k}{k}}B_k(\vartheta_i,2k)+k\frac{M_LR}{\gamma_{3-i}}\omega(\sigma,a)^{\frac{\vartheta_i+k\gamma_{3-i}}{k}}B_k(\vartheta_i,k\gamma_{3-i}+k)\right]+\frac{|\eta_i|}{|\Pi_i(b,\gamma_i)|}\\
&\quad+\frac{M_i|\nu_i|e^{\varrho\omega(b,a)}}{k\Gamma_k(\vartheta_i)|\Pi_i(b,\gamma_i)|}\left[\frac{k}{\vartheta_i}\omega(b,a)^{\frac{\vartheta_i}{k}}+kR\omega(b,a)^{\frac{\vartheta_i+k\gamma_i}{k}-1}B_k(\vartheta_i,k\gamma_i)\right.\\
&\quad\left.+kM_L\omega(b,a)^{\frac{\vartheta_i+k}{k}}B_k(\vartheta_i,2k)+k\frac{M_LR}{\gamma_{3-i}}\omega(b,a)^{\frac{\vartheta_i+k\gamma_{3-i}}{k}}B_k(\vartheta_i,k\gamma_{3-i}+k)\right],\qquad i=1,2.
\end{aligned}$$

This implies that

$$\|\mathcal{A}_i(\mathfrak{u}_1,\mathfrak{u}_2)\|_{\gamma_i}\le\widehat{R}_i,\qquad i=1,2,$$

where

$$\begin{aligned}
\widehat{R}_i:=&\frac{M_ie^{\varrho\omega(b,a)}}{\Gamma_k(\vartheta_i)}\left[\frac{1}{\vartheta_i}\omega(b,a)^{\frac{\vartheta_i}{k}+1-\gamma_i}+R\omega(b,a)^{\frac{\vartheta_i}{k}}B_k(\vartheta_i,k\gamma_i)\right.\\
&\left.+M_L\omega(b,a)^{\frac{\vartheta_i+2k-k\gamma_i}{k}}B_k(\vartheta_i,2k)+\frac{M_LR}{\gamma_{3-i}}\omega(b,a)^{\frac{\vartheta_i+k+k(\gamma_{3-i}-\gamma_i)}{k}}B_k(\vartheta_i,k\gamma_{3-i}+k)\right]
\end{aligned}$$

$$+\frac{|\eta_i|}{|\Pi_i(b,\gamma_i)|}+\frac{M_i|\nu_i|e^{\varrho\omega(b,a)}}{\Gamma_k(\vartheta_i)|\Pi_i(b,\gamma_i)|}\Bigg[\frac{1}{\vartheta_i}\omega(b,a)^{\frac{\vartheta_i}{k}}+R\omega(b,a)^{\frac{\vartheta_i+k\gamma_i}{k}-1}B_k(\vartheta_i,k\gamma_i)$$
$$+M_L\omega(b,a)^{\frac{\vartheta_i+k}{k}}B_k(\vartheta_i,2k)+\frac{M_LR}{\gamma_{3-i}}\omega(b,a)^{\frac{\vartheta_i+k\gamma_{3-i}}{k}}B_k(\vartheta_i,k\gamma_{3-i}+k)\Bigg],\qquad i=1,2.$$

Then, we get

$$R<\widehat{R}_i,\quad i=1,2. \tag{3.11}$$

Observe that $\widehat{R}_i$ is an affine function of R: All the terms of $\widehat{R}_i$ which do *not* carry the factor R (namely those involving $1/\vartheta_i$, $M_LB_k(\vartheta_i,2k)$ and $|\eta_i|/|\Pi_i(b,\gamma_i)|$) are constants independent of R, so that $\widehat{R}_i/R\to c_i$ as $R\to\infty$, where c_i is exactly the constant appearing on the right-hand side of (H5). In other words, under (H5) the operator $\mathcal{A}_i$ grows sub-linearly with respect to R, which is precisely what guarantees the existence of a ball $\mathbb{B}_R$ that is mapped into itself. Formally, dividing both sides of (3.11) by R and letting $R\to\infty$, we obtain

$$1<\frac{M_ie^{\varrho\omega(b,a)}}{\Gamma_k(\vartheta_i)}\Bigg(\omega(b,a)^{\frac{\vartheta_i}{k}}B_k\left(\vartheta_i,k\gamma_i\right)+\frac{M_L}{\gamma_{3-i}}\omega(b,a)^{\frac{\vartheta_i+k+k(\gamma_{3-i}-\gamma_i)}{k}}B_k\left(\vartheta_i,k\gamma_{3-i}+k\right)$$
$$+\frac{|\nu_i|\omega(b,a)^{\frac{\vartheta_i+k\gamma_i}{k}-1}}{|\Pi_i(b,\gamma_i)|}B_k\left(\vartheta_i,k\gamma_i\right)+\frac{M_L|\nu_i|\omega(b,a)^{\frac{\vartheta_i+k\gamma_{3-i}}{k}}}{\gamma_{3-i}|\Pi_i(b,\gamma_i)|}B_k\left(\vartheta_i,k\gamma_{3-i}+k\right)\Bigg),\ i=1,2,$$

which contradicts hypothesis (H5). Thus there is some $R\in\mathbb{N}$ such that $\mathcal{A}(\mathbb{B}_R)\subseteq\mathbb{B}_R$, that is, $\mathcal{A}:\mathbb{B}_R\to\mathbb{B}_R$.

Step 3. The equicontinuity of $\mathcal{A}(\mathbb{B}_R)$.

Using the fact that $0<e^{-\varrho\omega(\sigma_1,s)}\le 1$, for $a<s<\sigma_1$ and $a<\sigma_1<\sigma_2\le b$, we have

$$\begin{aligned}\left|\mathcal{K}_{-\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma_2,s)-\mathcal{K}_{-\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma_1,s)\right|&=e^{-\varrho\omega(\sigma_1,s)}\left|e^{-\varrho\omega(\sigma_2,\sigma_1)}\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}-\omega(\sigma_1,s)^{\frac{\vartheta_i}{k}-1}\right|\\&\le\omega(\sigma_1,s)^{\frac{\vartheta_i}{k}-1}-e^{-\varrho\omega(\sigma_2,\sigma_1)}\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1},\ i=1,2.\end{aligned} \tag{3.12}$$

Meanwhile, for all $(\mathfrak{u}_1,\mathfrak{u}_2)\in\mathbb{B}_R$, and $a\le\sigma_1\le\sigma_2\le b$, we obtain

$$\begin{aligned}&\|\omega(\sigma_2,a)^{1-\gamma_i}\mathcal{A}_i(\mathfrak{u}_1(\sigma_2),\mathfrak{u}_2(\sigma_2))-\omega(\sigma_1,a)^{1-\gamma_i}\mathcal{A}_i(\mathfrak{u}_1(\sigma_1),\mathfrak{u}_2(\sigma_1))\|\\&\le I_{0,i}+I_{1,i}+I_{2,i}+I_{3,i},\quad i=1,2,\end{aligned} \tag{3.13}$$

where,

$$\begin{aligned}I_{0,i}&=\frac{\omega(\sigma_2,a)^{1-\gamma_i}}{k\Gamma_k(\vartheta_i)}\int_{\sigma_1}^{\sigma_2}\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma_2,s)\|g_i(s,\mathfrak{u}_i(s),(\mathcal{H}\mathfrak{u}_{3-i})(s))\|ds,\ i=1,2,\\I_{1,i}&=\frac{\left(\omega(\sigma_2,a)^{1-\gamma_i}-\omega(\sigma_1,a)^{1-\gamma_i}\right)}{k\Gamma_k(\vartheta_i)}\int_{a}^{\sigma_1}\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma_2,s)\|g_i(s,\mathfrak{u}_i(s),(\mathcal{H}\mathfrak{u}_{3-i})(s))\|ds,\ i=1,2,\\I_{2,i}&=\frac{\omega(\sigma_1,a)^{1-\gamma_i}}{k\Gamma_k(\vartheta_i)}\int_{a}^{\sigma_1}\left|\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma_2,s)-\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma_1,s)\right|\varphi'(s)\|g_i(s,\mathfrak{u}_i(s),(\mathcal{H}\mathfrak{u}_{3-i})(s))\|ds,\ i=1,2,\end{aligned} \tag{3.14}$$

and

$$I_{3,i} = \frac{\left|e^{-\varrho\omega(\sigma_2,a)} - e^{-\varrho\omega(\sigma_1,a)}\right|}{|\Pi_i(b,\gamma_i)|} \times \left\|\eta_i - \frac{\nu_i}{k\Gamma_k(\vartheta_i)}\int_a^b \varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-\gamma_i}(b,s)g_i(s,\mathfrak{u}_i(s),(\mathcal{H}\mathfrak{u}_{3-i})(s))ds\right\|, \ i=1,2. \tag{3.15}$$

Now, combining (H3) and the fact $0 < e^{-\varrho\omega(\sigma_1,s)} \leq 1$, for $0 < s < \sigma_1$, and (3.6), we get

$$\begin{aligned} I_{0,i} &\leq \frac{\omega(\sigma_2,a)^{1-\gamma_i}}{k\Gamma_k(\vartheta_i)}\int_{\sigma_1}^{\sigma_2}\varphi'(s)\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}M_i\left(1+\|\mathfrak{u}_i(s)\|+\|(\mathcal{H}\mathfrak{u}_{3-i})(s)\|\right)ds \\ &\leq \frac{\omega(\sigma_2,a)^{1-\gamma_i}}{k\Gamma_k(\vartheta_i)}\int_{\sigma_1}^{\sigma_2}\varphi'(s)\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}M_i\Big[1+R\omega(s,a)^{\gamma_i-1} \\ &\quad +M_L\Big(\omega(s,a)+R\frac{\omega(s,a)^{\gamma_{3-i}}}{\gamma_{3-i}}\Big)\Big]ds, \quad i=1,2, \end{aligned} \tag{3.16}$$

which after simple calculations gives:

$$\frac{1}{k\Gamma_k(\vartheta_i)}\int_{\sigma_1}^{\sigma_2}\varphi'(s)\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}ds = \frac{\omega(\sigma_2,\sigma_1)^{\frac{\vartheta_i}{k}}}{\Gamma_k(\vartheta_i+k)}, \ i=1,2, \tag{3.17}$$

$$\begin{aligned} \frac{1}{k\Gamma_k(\vartheta_i)}\int_{\sigma_1}^{\sigma_2}\varphi'(s)\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)ds &\leq \frac{\omega(b,a)}{k\Gamma_k(\vartheta_i)}\int_{\sigma_1}^{\sigma_2}\varphi'(s)\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}ds \\ &\leq \frac{\omega(b,a)}{\Gamma_k(\vartheta_i+k)}\omega(\sigma_2,\sigma_1)^{\frac{\vartheta_i}{k}}, \ i=1,2, \end{aligned} \tag{3.18}$$

$$\int_{\sigma_1}^{\sigma_2}\varphi'(s)\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_i-1}ds = \Theta_{\frac{\vartheta_i}{k},1-\gamma_i}(\sigma_1,\sigma_2,\sigma_2), \ i=1,2, \tag{3.19}$$

and

$$\int_{\sigma_1}^{\sigma_2}\varphi'(s)\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_{3-i}-1}ds = \Theta_{\frac{\vartheta_i}{k},1-\gamma_{3-i}}(\sigma_1,\sigma_2,\sigma_2), \ i=1,2, \tag{3.20}$$

where

$$\Theta_{\frac{\vartheta_i}{k},1-\gamma_j}(\sigma_1,\sigma_2,\sigma_2) = \int_{\sigma_1}^{\sigma_2}\varphi'(s)\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{-(1-\gamma_j)}ds, \quad i,j=1,2.$$

From (3.16)-(3.20) and using [16, Remark 2.2], we obtain

$$I_{0,i} \longrightarrow 0 \quad \text{as} \quad \sigma_2 \longrightarrow \sigma_1, \quad i=1,2. \tag{3.21}$$

Now, in view of (H3), (3.12) and (3.6), we get

$$\begin{aligned} I_{2,i} &\leq \frac{\omega(\sigma_1,a)^{1-\gamma_i}}{k\Gamma_k(\vartheta_i)}\int_a^{\sigma_1}\left(\omega(\sigma_1,s)^{\frac{\vartheta_i}{k}-1} - e^{-\varrho\omega(\sigma_2,\sigma_1)}\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}\right)\varphi'(s)M_i \\ &\quad \times\left[1+R\omega(s,a)^{\gamma_i-1}+M_L\Big(\omega(s,a)+R\frac{\omega(s,a)^{\gamma_{3-i}}}{\gamma_{3-i}}\Big)\right]ds, \quad i=1,2. \end{aligned} \tag{3.22}$$

By the same techniques, we get

$$\begin{aligned}&\frac{1}{k\Gamma_k(\vartheta_i)}\int_a^{\sigma_1}\left(\omega(\sigma_1,s)^{\frac{\vartheta_i}{k}-1}-e^{-\varrho\omega(\sigma_2,\sigma_1)}\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}\right)\varphi'(s)ds\\&=\frac{1}{\Gamma_k(\vartheta_i+k)}\left(\omega(\sigma_1,a)^{\frac{\vartheta_i}{k}}+e^{-\varrho\omega(\sigma_2,\sigma_1)}\left(\omega(\sigma_2,\sigma_1)^{\frac{\vartheta_i}{k}}-\omega(\sigma_2,a)^{\frac{\vartheta_i}{k}}\right)\right),\quad i=1,2,\end{aligned}\tag{3.23}$$

$$\begin{aligned}&\frac{1}{k\Gamma_k(\vartheta_i)}\int_a^{\sigma_1}\left(\omega(\sigma_1,s)^{\frac{\vartheta_i}{k}-1}-e^{-\varrho\omega(\sigma_2,\sigma_1)}\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}\right)\varphi'(s)\omega(s,a)ds\\&=\frac{\omega(b,a)}{\Gamma_k(\vartheta_i+k)}\left(\omega(\sigma_1,a)^{\frac{\vartheta_i}{k}}+e^{-\varrho\omega(\sigma_2,\sigma_1)}\left(\omega(\sigma_2,\sigma_1)^{\frac{\vartheta_i}{k}}-\omega(\sigma_2,a)^{\frac{\vartheta_i}{k}}\right)\right),\quad i=1,2,\end{aligned}\tag{3.24}$$

$$\begin{aligned}&\int_a^{\sigma_1}\left(\omega(\sigma_1,s)^{\frac{\vartheta_i}{k}-1}-e^{-\varrho\omega(\sigma_2,\sigma_1)}\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}\right)\varphi'(s)\omega(s,a)^{\gamma_i-1}ds\\&=\Theta_{\frac{\vartheta_i}{k},1-\gamma_i}(a,\sigma_1,\sigma_1)-e^{-\varrho\omega(\sigma_2,\sigma_1)}\Theta_{\frac{\vartheta_i}{k},1-\gamma_i}(a,\sigma_1,\sigma_2),\ i=1,2,\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}&\int_a^{\sigma_1}\left(\omega(\sigma_1,s)^{\frac{\vartheta_i}{k}-1}-e^{-\varrho\omega(\sigma_2,\sigma_1)}\omega(\sigma_2,s)^{\frac{\vartheta_i}{k}-1}\right)\varphi'(s)\omega(s,a)^{\gamma_{3-i}-1}ds\\&=\Theta_{\frac{\vartheta_i}{k},1-\gamma_{3-i}}(a,\sigma_1,\sigma_1)-e^{-\varrho\omega(\sigma_2,\sigma_1)}\Theta_{\frac{\vartheta_i}{k},1-\gamma_{3-i}}(a,\sigma_1,\sigma_2),\ i=1,2.\end{aligned}\tag{3.26}$$

Inequalities (3.22) and (3.26), together with [16, Remark 2.2], give

$$I_{2,i}\longrightarrow 0\quad\text{as}\quad\sigma_2\longrightarrow\sigma_1,\quad i=1,2.\tag{3.27}$$

From (3.13), (3.14), (3.15), (3.21) and (3.27), we obtain

$$\|\omega(\sigma_2,a)^{1-\gamma_i}\mathcal{A}_i(\mathfrak{u}_1(\sigma_2),\mathfrak{u}_2(\sigma_2))-\omega(\sigma_1,a)^{1-\gamma_i}\mathcal{A}_i(\mathfrak{u}_1(\sigma_1),\mathfrak{u}_2(\sigma_1))\|\longrightarrow 0\quad\text{as}\quad\sigma_2\longrightarrow\sigma_1,\ i=1,2.$$

Therefore, the equicontinuity of $\mathcal{A}(\mathbb{B}_R)$ is established.

Step 4. The Mönch condition (2.5) is satisfied.

Let $\mathbb{J}^1\times\mathbb{J}^2$ be a countable subset of $\mathbb{B}_R$ and $\mathbb{J}^1\times\mathbb{J}^2\subseteq\operatorname{conv}\left(\mathcal{A}_i(\mathbb{J}^1\times\mathbb{J}^2)\cup\{(0,0)\}\right)$, $i=1,2$. For every bounded subset $D^1\times D^2\subset C_{\gamma_1,\gamma_2}(\mathcal{L},\mathbb{K})$, we define the MNC as

$$\Upsilon_{\gamma_1,\gamma_2}(D^1\times D^2)=\Upsilon_{\gamma_1}(D^1)+\Upsilon_{\gamma_2}(D^2),\tag{3.28}$$

where

$$\Upsilon_{\gamma_1}(D^i)=\sup_{\sigma\in\mathcal{L}}\frac{\Upsilon(D^i_{\gamma_i}(\sigma))}{e^{\aleph\sigma}};\quad\aleph>0,\quad i=1,2.\tag{3.29}$$

The exponential weight $e^{-\aleph\sigma}$ in (3.29) means that $\Upsilon_{\gamma_1,\gamma_2}$ is a *Bielecki-type* measure of noncompactness, i.e. the MNC counterpart of the classical Bielecki norm. Its role is purely technical: The free parameter $\aleph>0$ can be chosen large enough to make the constants $\Lambda_i(\aleph)$, $\Delta_i(\aleph)$ below arbitrarily small, which replaces a Gronwall-type iteration argument by a single contraction-like estimate and thus considerably simplifies the verification of the Mönch condition.

By using Lemma 2.3 and [1, Example 1.5], $\Upsilon_{\gamma_1,\gamma_2}$ satisfies all properties of the Hausdorff MNC.

On one hand, let

$$\Lambda_i(\aleph) := \sup_{\sigma\in\mathcal{L}} \int_a^\sigma \varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_i-1}e^{-\aleph(\sigma-s)}ds,\ i=1,2, \tag{3.30}$$

$$\Lambda_{3-i}(\aleph) := \sup_{\sigma\in\mathcal{L}} \int_a^\sigma \varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_{3-i}-1}e^{-\aleph(\sigma-s)}ds,\ i=1,2, \tag{3.31}$$

$$\Delta_i(\aleph) := \sup_{\sigma\in\mathcal{L}} \int_a^b \varphi'(s)\omega(b,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_i-1}e^{-\aleph(\sigma-s)}ds,\ i=1,2, \tag{3.32}$$

and

$$\Delta_{3-i}(\aleph) := \sup_{\sigma\in\mathcal{L}} \int_a^b \varphi'(s)\omega(b,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_{3-i}-1}e^{-\aleph(\sigma-s)}ds,\ i=1,2. \tag{3.33}$$

From [16, Lemma 2.1], we get

$$\omega(\sigma,\cdot)^{\frac{\vartheta_i}{k}-1}\omega(\cdot,a)^{\gamma_j-1}\varphi'(\cdot)\in L^\infty(\mathcal{L},\mathbb{R}),\quad i,j=1,2.$$

Accordingly, one can select $\aleph$ to satisfy

$$\frac{2\Psi^*_{i,1}\mathcal{K}^{1-\gamma_i}_{-\varrho,\omega}(b,a)}{k\Gamma_k(\vartheta_i)}\left(\Lambda_i(\aleph)+\frac{|\nu_i|\Delta_i(\aleph)}{|\Pi_i(b,\gamma_i)|}\right)<\frac{1}{3},\quad i=1,2, \tag{3.34}$$

and

$$\frac{4\Psi^*_{i,2}\sigma^*_{3-i}\mathcal{K}^{2-\gamma_i}_{-\varrho,\omega}(b,a)}{k\Gamma_k(\vartheta_i)}\left(\Lambda_{3-i}(\aleph)+\frac{|\nu_i|\Delta_{3-i}(\aleph)}{|\Pi_i(b,\gamma_i)|}\right)<\frac{1}{3},\quad i=1,2. \tag{3.35}$$

Meanwhile, let $\{Z_1^n,Z_2^n\}_{n=1}^{+\infty}\subseteq\mathcal{A}(\mathbb{J}^1\times\mathbb{J}^2)$ be the countable set and

$$\Upsilon_{\gamma_i}(\{Z_i^n\}_{n=1}^{+\infty})=\Upsilon_{\gamma_i}\left(\mathcal{A}_i(\mathbb{J}^1\times\mathbb{J}^2)\right),\quad i=1,2. \tag{3.36}$$

Then, there exists $\{(\mathfrak{u}^{1;n},\mathfrak{u}^{2;n})\}_{n=1}^{+\infty}\subset\mathbb{J}^1\times\mathbb{J}^2$, such that

$$\begin{aligned} Z_i^n(\sigma)&=\mathcal{A}_i(\mathfrak{u}^{1,n}(\sigma),\mathfrak{u}^{2,n}(\sigma))\\ &={}^T_k\mathcal{I}^{\vartheta_i,\varrho;\varphi}_{a^+}g_i(s,\mathfrak{u}^{i,n}(\sigma),(\mathcal{H}\mathfrak{u}^{3-i,n})(\sigma))+\frac{\eta_i\mathcal{K}^{\gamma_i-1}_{\varrho,\omega}(\sigma,a)}{\Pi_i(b,\gamma_i)}\\ &\quad-\frac{\nu_i\mathcal{K}^{\gamma_i-1}_{\varrho,\omega}(\sigma,a)}{\Pi_i(b,\gamma_i)}\left({}^T_k\mathcal{I}^{\vartheta_i,\varrho;\varphi}_{a^+}g_i(s,\mathfrak{u}^{i,n}(\sigma),(\mathcal{H}\mathfrak{u}^{3-i,n})(\sigma))\right)(b),\quad i=1,2. \end{aligned} \tag{3.37}$$

For $n\geq1$, $\sigma\in\mathcal{L}$ and $\{(\mathfrak{u}^{1,n},\mathfrak{u}^{2,n})\}_{n=1}^{\infty}\subset C_{\gamma_1,\gamma_2}(\mathcal{L},\mathbb{K})$. Using the fact that $0<e^{-\varrho\omega(\sigma,s)}\leq1$, for $a<s<\sigma$ and using the properties of the MNC, one has

$$\begin{aligned} \Upsilon_{\gamma_i}(\{Z_i^n\}_{n=1}^{+\infty})&\leq\Upsilon_{\gamma_i}\left(\left\{{}^T_k\mathcal{I}^{\vartheta_i,\varrho;\varphi}_{a^+}g_i(\cdot,\mathfrak{u}^{i,n}(\cdot),(\mathcal{H}\mathfrak{u}^{3-i,n})(\cdot))\right\}_{n=1}^{+\infty}\right)\\ &\quad+\frac{|\nu_i|}{|\Pi_i(b,\gamma_i)|}\Upsilon_{\gamma_i}\left(\left\{\left({}^T_k\mathcal{I}^{\vartheta_i,\varrho;\varphi}_{a^+}g_i(\cdot,\mathfrak{u}^{i,n}(\cdot),(\mathcal{H}\mathfrak{u}^{3-i,n})(\cdot))\right)(b)\right\}_{n=1}^{+\infty}\right),\ i=1,2. \end{aligned} \tag{3.38}$$

We now proceed to estimate the quantity

$$\Upsilon_{\gamma_i}\left(\left\{{}^T_k\mathcal{I}^{\vartheta_i,\varrho;\varphi}_{a^+}g_i(\cdot,\mathfrak{u}^{i,n}(\cdot),(\mathcal{H}\mathfrak{u}^{3-i,n})(\cdot))\right\}_{n=1}^{+\infty}\right),\quad i=1,2$$

and

$$\Upsilon_{\gamma_i}\left(\left\{\left({}_k^T\mathcal{I}_{a^+}^{\vartheta_i,\varrho;\varphi}g_i(\cdot,\mathfrak{u}^{i,n}(\cdot),(\mathcal{H}\mathfrak{u}^{3-i,n})(\cdot))\right)(b)\right\}_{n=1}^{+\infty}\right),\quad i=1,2.$$

Firstly, using condition (H4) and Lemma 2.4, for each $\sigma\in\mathcal{L}$ and $s\le\sigma$, one has

$$\begin{aligned}
&\Upsilon\left(\left\{\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)}{k\Gamma_k(\vartheta_i)}g_i\big(s,\mathfrak{u}^{i,n}(s),(\mathcal{H}\mathfrak{u}^{3-i,n})(s)\big)\right\}_{n=1}^{\infty}\right)\\
&\le\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)}{k\Gamma_k(\vartheta_i)}\left[\Psi_{i,1}(s)\Upsilon\left(\{\mathfrak{u}^{i,n}(s)\}_{n=1}^{\infty}\right)+\Psi_{i,2}(s)\Upsilon\left(\{(\mathcal{H}\mathfrak{u}^{3-i,n})(s)\}_{n=1}^{\infty}\right)\right]\\
&\le\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)}{k\Gamma_k(\vartheta_i)}\left[\Psi_{i,1}(s)\Upsilon\left(\{\mathfrak{u}^{i,n}(s)\}_{n=1}^{\infty}\right)\right.\\
&\quad\left.+\Psi_{i,2}(s)\Upsilon\left(\int_a^s L\big(s,\tau,\{\mathfrak{u}^{3-i,n}(\tau)\}_{n=1}^{\infty}\big)\varphi'(\tau)\,d\tau\right)\right]\\
&\le\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)}{k\Gamma_k(\vartheta_i)}\left[\Psi_{i,1}(s)\Upsilon\left(\{\mathfrak{u}^{i,n}(s)\}_{n=1}^{\infty}\right)\right.\\
&\quad\left.+2\Psi_{i,2}(s)\int_a^s\Upsilon\left(L\big(s,\tau,\{\mathfrak{u}^{3-i,n}(\tau)\}_{n=1}^{\infty}\big)\varphi'(\tau)\,d\tau\right)\right]\\
&\le\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)}{k\Gamma_k(\vartheta_i)}\left[\Psi_{i,1}(s)\Upsilon\left(\{\mathfrak{u}^{i,n}(s)\}_{n=1}^{\infty}\right)\right.\\
&\quad\left.+2\Psi_{i,2}(s)\int_a^s\sigma_{3-i}(\tau)\Upsilon\left(\{\mathfrak{u}^{3-i,n}(\tau)\}_{n=1}^{\infty}\right)\varphi'(\tau)\,d\tau\right],\qquad i=1,2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Upsilon\left(\left\{\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)}{k\Gamma_k(\vartheta_i)}g_i\big(s,\mathfrak{u}^{i,n}(s),(\mathcal{H}\mathfrak{u}^{3-i,n})(s)\big)\right\}_{n=1}^{\infty}\right)\\
&\le\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)}{k\Gamma_k(\vartheta_i)}\left[\Psi_{i,1}(s)\Upsilon\left(\{\mathfrak{u}^{i,n}(s)\}_{n=1}^{\infty}\right)\right.\\
&\quad\left.+2\Psi_{i,2}(s)\sigma_{3-i}(s)\Upsilon\left(\{\mathfrak{u}^{3-i,n}(s)\}_{n=1}^{\infty}\right)\int_a^s\varphi'(\tau)\,d\tau\right]\\
&\le\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)}{k\Gamma_k(\vartheta_i)}\left[\Psi_{i,1}(s)\Upsilon\left(\{\mathfrak{u}^{i,n}(s)\}_{n=1}^{\infty}\right)\right.\\
&\quad\left.+2\Psi_{i,2}(s)\sigma_{3-i}(s)\Upsilon\left(\{\mathfrak{u}^{3-i,n}(s)\}_{n=1}^{\infty}\right)\omega(s,a)\right]\\
&\le\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)}{k\Gamma_k(\vartheta_i)}\left[\Psi_{i,1}(s)\omega(s,a)^{\gamma_i-1}\Upsilon\left(\{\mathfrak{u}_{\gamma_i}^{i,n}(s)\}_{n=1}^{\infty}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+ 2\Psi_{i,2}(s)\sigma_{3-i}(s)\omega(s,a)^{\gamma_{3-i}}\Upsilon\left(\{\mathfrak{u}_{\gamma_i}^{3-i,n}(s)\}_{n=1}^{\infty}\right)\Bigg]\\
\leq&\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)e^{\aleph s}}{k\Gamma_k(\vartheta_i)}\Bigg[\Psi_{i,1}(s)\omega(s,a)^{\gamma_i-1}\sup_{0\leq s\leq\sigma}e^{-\aleph s}\Upsilon\left(\{\mathfrak{u}_{\gamma_i}^{i,n}(s)\}_{n=1}^{\infty}\right)\\
&+ 2\Psi_{i,2}(s)\sigma_{3-i}(s)\omega(s,a)^{\gamma_{3-i}}\sup_{0\leq s\leq\sigma}e^{-\aleph s}\Upsilon\left(\{\mathfrak{u}_{\gamma_i}^{3-i,n}(s)\}_{n=1}^{\infty}\right)\Bigg]\\
\leq&\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)e^{\aleph s}}{k\Gamma_k(\vartheta_i)}\Bigg[\Psi_{i,1}(s)\omega(s,a)^{\gamma_i-1}\Upsilon_{\gamma_i}\left(\{\mathfrak{u}^{i,n}\}_{n=1}^{\infty}\right)\\
&+ 2\Psi_{i,2}(s)\sigma_{3-i}(s)\omega(s,a)^{\gamma_{3-i}}\Upsilon_{\gamma_i}\left(\{\mathfrak{u}^{3-i,n}\}_{n=1}^{\infty}\right)\Bigg],\qquad i=1,2.
\end{aligned}
$$

Consequently, Lemma 2.4 implies that

$$
\begin{aligned}
&\Upsilon\left(\left\{\int_a^{\sigma}\frac{\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)}{k\Gamma_k(\vartheta_i)}g_i\big(s,\mathfrak{u}^{i,n}(s),(\mathcal{H}\mathfrak{u}^{3-i,n})(s)\big)\,ds\right\}_{n=1}^{\infty}\right)\\
\leq&\Upsilon_{\gamma_i}\left(\{\mathfrak{u}^{i,n}\}_{n=1}^{\infty}\right)\frac{2\Psi_{i,1}^*}{k\Gamma_k(\vartheta_i)}\int_a^{\sigma}\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)\omega(s,a)^{\gamma_i-1}e^{\aleph s}\,ds\\
&+\Upsilon_{\gamma_{3-i}}\left(\{\mathfrak{u}^{3-i,n}\}_{n=1}^{\infty}\right)\frac{4\Psi_{i,2}^*\sigma_{3-i}^*}{k\Gamma_k(\vartheta_i)}\int_a^{\sigma}\varphi'(s)\mathcal{K}_{\varrho,\omega}^{\frac{\vartheta_i}{k}-1}(\sigma,s)\omega(s,a)^{\gamma_{3-i}}e^{\aleph s}\,ds,\qquad i=1,2.
\end{aligned}
$$

Using (3.7), we get

$$
\begin{aligned}
&\Upsilon\left(\left\{\omega(\sigma,a)^{1-\gamma_i}{}_k^T\mathcal{I}_{a^+}^{\vartheta_i,\varrho;\varphi}g_i\big(\sigma,\mathfrak{u}^{i,n}(\sigma),(\mathcal{H}\mathfrak{u}^{3-i,n})(\sigma)\big)\right\}_{n=1}^{\infty}\right)\\
\leq&\Upsilon_{\gamma_i}\left(\{\mathfrak{u}^{i,n}\}_{n=1}^{\infty}\right)\frac{2\Psi_{i,1}^*\mathcal{K}_{-\varrho,\omega}^{1-\gamma_i}(b,a)}{k\Gamma_k(\vartheta_i)}\int_a^{\sigma}\varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_i-1}e^{\aleph s}\,ds\\
&+\Upsilon_{\gamma_{3-i}}\left(\{\mathfrak{u}^{3-i,n}\}_{n=1}^{\infty}\right)\frac{4\Psi_{i,2}^*\sigma_{3-i}^*\mathcal{K}_{-\varrho,\omega}^{1-\gamma_i}(b,a)}{k\Gamma_k(\vartheta_i)}\int_a^{\sigma}\varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_{3-i}}e^{\aleph s}\,ds\\
\leq&\Upsilon_{\gamma_i}\left(\{\mathfrak{u}^{i,n}\}_{n=1}^{\infty}\right)\frac{2\Psi_{i,1}^*\mathcal{K}_{-\varrho,\omega}^{1-\gamma_i}(b,a)}{k\Gamma_k(\vartheta_i)}\int_a^{\sigma}\varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_i-1}e^{\aleph s}\,ds\\
&+\Upsilon_{\gamma_{3-i}}\left(\{\mathfrak{u}^{3-i,n}\}_{n=1}^{\infty}\right)\frac{4\Psi_{i,2}^*\sigma_{3-i}^*\mathcal{K}_{-\varrho,\omega}^{2-\gamma_i}(b,a)}{k\Gamma_k(\vartheta_i)}\int_a^{\sigma}\varphi'(s)\omega(\sigma,s)^{\frac{\vartheta_i}{k}-1}\omega(s,a)^{\gamma_{3-i}-1}e^{\aleph s}\,ds,\quad i=1,2.
\end{aligned}
$$

Multiplying both sides by $e^{-\aleph\sigma}$ and taking $\sup_{\sigma\in\mathcal{L}}$, one obtains

$$
\begin{aligned}
&\Upsilon_{\gamma_i}\left(\left\{{}_k^T\mathcal{I}_{a^+}^{\vartheta_i,\varrho;\varphi}g_i(\cdot,\mathfrak{u}^{i,n}(\cdot),(\mathcal{H}\mathfrak{u}^{3-i,n})(\cdot))\right\}_{n=1}^{\infty}\right)\\
\leq&\sup_{\sigma\in\mathcal{L}}e^{-\aleph\sigma}\Upsilon\left(\left\{\omega(\sigma,a)^{1-\gamma_i}{}_k^T\mathcal{I}_{a^+}^{\vartheta_i,\varrho;\varphi}g_i(\sigma,\mathfrak{u}^{i,n}(\sigma),(\mathcal{H}\mathfrak{u}^{3-i,n})(\sigma))\right\}_{n=1}^{\infty}\right)\\
\leq&\Upsilon_{\gamma_i}\left(\{\mathfrak{u}^{i,n}\}_{n=1}^{\infty}\right)\frac{2\Psi_{i,1}^*\mathcal{K}_{-\varrho,\omega}^{1-\gamma_i}(b,a)}{k\Gamma_k(\vartheta_i)}\Lambda_i(\aleph)
\end{aligned}
$$

$$+\Upsilon_{\gamma_{3-i}}\left(\{\mathfrak{u}^{3-i,n}\}_{n=1}^{\infty}\right)\frac{4\Psi_{i,2}^{*}\sigma_{3-i}^{*}\mathcal{K}_{-\varrho,\omega}^{2-\gamma_i}(b,a)}{k\Gamma_k(\vartheta_i)}\Lambda_{3-i}(\aleph),\quad i=1,2, \tag{3.39}$$

where $\Lambda_i(\aleph)$ and $\Lambda_{3-i}(\aleph)$, $i=1,2$ are defined in (3.30) and (3.31) respectively.

Using the same techniques, we get

$$\begin{aligned}
&\frac{|\nu_i|}{|\Pi_i(b,\gamma_i)|}\Upsilon_{\gamma_i}\left(\left\{\left({}_k^T\mathcal{I}_{a^+}^{\vartheta_i,\varrho;\varphi}g_i(\cdot,\mathfrak{u}^{i,n}(\cdot),(\mathcal{H}\mathfrak{u}^{3-i,n})(\cdot))\right)(b)\right\}_{n=1}^{+\infty}\right)\\
&\leq\Upsilon_{\gamma_i}\left(\{\mathfrak{u}^{i,n}\}_{n=1}^{\infty}\right)\frac{2|\nu_i|\Psi_{i,1}^{*}\mathcal{K}_{-\varrho,\omega}^{1-\gamma_i}(b,a)}{k\Gamma_k(\vartheta_i)|\Pi_i(b,\gamma_i)|}\Delta_i(\aleph)\\
&\quad+\Upsilon_{\gamma_{3-i}}\left(\{\mathfrak{u}^{3-i,n}\}_{n=1}^{\infty}\right)\frac{4|\nu_i|\Psi_{i,2}^{*}\sigma_{3-i}^{*}\mathcal{K}_{-\varrho,\omega}^{2-\gamma_i}(b,a)}{k\Gamma_k(\vartheta_i)|\Pi_i(b,\gamma_i)|}\Delta_{3-i}(\aleph),\quad i=1,2,
\end{aligned} \tag{3.40}$$

where $\Delta_i(\aleph)$ and $\Delta_{3-i}(\aleph)$, $i=1,2$ are defined in (3.32) and (3.33) respectively.

Combining (3.38), (3.39) and (3.40) yields (for $i=1,2$)

$$\begin{aligned}
\Upsilon_{\gamma_i}(\{Z_i^n\}_{n=1}^{+\infty})&\leq\Upsilon_{\gamma_i}\left(\{\mathfrak{u}^{i,n}\}_{n=1}^{\infty}\right)\frac{2\Psi_{i,1}^{*}\mathcal{K}_{-\varrho,\omega}^{1-\gamma_i}(b,a)}{k\Gamma_k(\vartheta_i)}\left(\Lambda_i(\aleph)+\frac{|\nu_i|\Delta_i(\aleph)}{|\Pi_i(b,\gamma_i)|}\right)\\
&\quad+\Upsilon_{\gamma_{3-i}}\left(\{\mathfrak{u}^{3-i,n}\}_{n=1}^{\infty}\right)\frac{4\Psi_{i,2}^{*}\sigma_{3-i}^{*}\mathcal{K}_{-\varrho,\omega}^{2-\gamma_i}(b,a)}{k\Gamma_k(\vartheta_i)}\left(\Lambda_{3-i}(\aleph)+\frac{|\nu_i|\Delta_{3-i}(\aleph)}{|\Pi_i(b,\gamma_i)|}\right).
\end{aligned}$$

From the inequalities (3.34) and (3.35), we have

$$\Upsilon_{\gamma_i}(\{Z_i^n\}_{n=1}^{+\infty})\leq\frac{1}{3}\Upsilon_{\gamma_i}\left(\{\mathfrak{u}^{i,n}\}_{n=1}^{\infty}\right)+\frac{1}{3}\Upsilon_{\gamma_{3-i}}\left(\{\mathfrak{u}^{3-i,n}\}_{n=1}^{\infty}\right),\ i=1,2. \tag{3.41}$$

And by properties of MNC, we have

$$\begin{aligned}
\Upsilon_{\gamma_i}(\{\mathfrak{u}^{i,n}\}_{n=1}^{+\infty})&\leq\Upsilon_{\gamma_i}\left(\mathbb{J}^1\times\mathbb{J}^2\right)\\
&\leq\Upsilon_{\gamma_i}\left(\mathrm{conv}\left(\mathcal{A}_i(\mathbb{J}^1\times\mathbb{J}^2)\cup\{(0,0)\}\right)\right)\\
&=\Upsilon_{\gamma_i}\left(\{Z_i^n\}_{n=1}^{+\infty}\right),\quad i=1,2.
\end{aligned} \tag{3.42}$$

Inequalities (3.41) and (3.42), together with (3.28), give

$$\begin{aligned}
\Upsilon_{\gamma_1,\gamma_2}(\{(Z_1^n,Z_2^n)\}_{n=1}^{+\infty})&=\Upsilon_{\gamma_1}(\{Z_1^n\}_{n=1}^{+\infty})+\Upsilon_{\gamma_2}(\{Z_2^n\}_{n=1}^{+\infty})\\
&\leq\frac{2}{3}\Upsilon_{\gamma_1}\left(\{\mathfrak{u}^{1,n}\}_{n=1}^{\infty}\right)+\frac{2}{3}\Upsilon_{\gamma_2}\left(\{\mathfrak{u}^{2,n}\}_{n=1}^{\infty}\right)\\
&\leq\frac{2}{3}\Upsilon_{\gamma_1,\gamma_2}(\{(Z_1^n,Z_2^n)\}_{n=1}^{+\infty}),
\end{aligned}$$

which implies that $\Upsilon_{\gamma_1,\gamma_2}(\mathbb{J}^1\times\mathbb{J}^2)=(0,0)$ and entails the relative compactness of $\mathbb{J}^1\times\mathbb{J}^2$ in $\mathbb{B}_R$.

Accordingly, the implication (2.5) holds.

Hence, by Theorem 2.1, there is a fixed point $(\mathfrak{u}_1,\mathfrak{u}_2)$ of operator $\mathcal{A}$, which is a solution of the problem (1.1) in $C_{\gamma_1,\gamma_2}(\mathcal{L},\mathbb{K})$. □

Remark 3.2. Theorem 3.1 contains several known results as particular cases, which makes the claimed generalization concrete.

(i) For $k = 1$, $\varrho = 0$ and $\varphi(\sigma) = \sigma$, the operator ${}^{T}_{k}\mathcal{D}^{\vartheta_i,\kappa_i,\varrho;\varphi}_{a^+}$ reduces to the classical Hilfer derivative $\mathcal{D}^{\vartheta_i,\kappa_i}_{a^+}$, γ_i reduces to $\vartheta_i + \kappa_i(1-\vartheta_i)$, and (1.1) becomes a coupled version of the boundary value problem studied in [24]; in the scalar (decoupled) case $g_1 = g_2$, $\mathfrak{u}_1 = \mathfrak{u}_2$, one recovers [24, Theorem 3.1] verbatim.

(ii) For $k = 1$, $\varrho = 0$, $\varphi(\sigma) = \sigma$ and $\kappa_i = 1$, one obtains the Caputo integro-differential setting of [25].

(iii) For $k = 1$ and $\varrho = 0$ with a general φ, Theorem 3.1 reduces to the φ-Hilfer result of [29], while for $\kappa_i = 0$ it covers the φ-tempered Riemann–Liouville systems of [32].

(iv) When $\dim \mathbb{K} < \infty$, hypothesis (H4) is automatically satisfied with $\Psi_{i,j} \equiv 0$ and $\sigma_i \equiv 0$, since every bounded set is relatively compact; the Mönch condition then holds trivially and Theorem 3.1 reduces to a classical Schauder-type existence result.

4. Example

In this section, one example is provided to show the applicability of the analytical achievements of the paper. Let

$$\begin{cases} {}^{T}_{k}\mathcal{D}^{\vartheta_1,\kappa_1,\varrho;e^\sigma}_{a^+}\mathfrak{u}_1(\sigma) = \left\{ \dfrac{1}{2\mathfrak{Z}_1(b)e^\sigma} + \dfrac{|\mathfrak{u}^{1,n}|}{2\mathfrak{Z}_1(b)e^\sigma} + \dfrac{\tan^{-1}(|\mathcal{H}\mathfrak{u}^{2,n}|)}{2\mathfrak{Z}_1(b)e^\sigma} \right\}_{n\geq 1}, \quad \sigma \in (a,b], \\ \quad {}^{T}_{k}\mathcal{D}^{\vartheta_2,\kappa_2,\varrho;e^\sigma}_{a^+}\mathfrak{u}_2(\sigma) \\ = \dfrac{1+\sigma}{3(1+b)\mathfrak{Z}_2(b)} \left\{ 1 + \log_e(|\mathcal{H}\mathfrak{u}^{1,n}|+1) + \dfrac{2}{\pi}\tan^{-1}(\sigma)\sin(\mathfrak{u}^{2,n}) \right\}_{n\geq 1}, \quad \sigma \in (a,b], \\ \mu_1 \left({}^{T}_{k}\mathcal{I}^{k(1-\gamma_1),\varrho;e^\sigma}_{a^+}\mathfrak{u}_1 \right)(a^+) + \nu_1 \left({}^{T}_{k}\mathcal{I}^{k(1-\gamma_1),\varrho;e^\sigma}_{0^+}\mathfrak{u}_1 \right)(b) = \eta_1, \\ \mu_2 \left({}^{T}_{k}\mathcal{I}^{k(1-\gamma_2),\varrho;e^\sigma}_{a^+}\mathfrak{u}_2 \right)(a^+) + \nu_2 \left({}^{T}_{k}\mathcal{I}^{k(1-\gamma_2),\varrho;e^\sigma}_{0^+}\mathfrak{u}_2 \right)(b) = \eta_2, \end{cases} \tag{4.1}$$

where, for $i = 1, 2$, $\mathfrak{Z}_i(b)$ denotes the finite positive constant, depending only on the endpoints a, b and on the fractional parameters $\vartheta_i, \gamma_i, k, \varrho, \nu_i$, defined by

$$\mathfrak{Z}_i(b) = \frac{e^{\varrho(e^b-e^a)}}{\Gamma_k(\vartheta_i)} \Bigg(\left(e^b-e^a\right)^{\frac{\vartheta_i}{k}} B_k\left(\vartheta_i, k\gamma_i\right) + \frac{e^b}{\gamma_{3-i}} \left(e^b-e^a\right)^{\frac{\vartheta_i+k+k(\gamma_{3-i}-\gamma_i)}{k}} B_k\left(\vartheta_i, k\gamma_{3-i}+k\right)$$
$$+ \frac{|\nu_i|\left(e^b-e^a\right)^{\frac{\vartheta_i+k\gamma_i}{k}-1}}{|\Pi_i(b,\gamma_i)|} B_k\left(\vartheta_i, k\gamma_i\right) + \frac{e^b|\nu_i|\left(e^b-e^a\right)^{\frac{\vartheta_i+k\gamma_{3-i}}{k}}}{\gamma_{3-i}|\Pi_i(b,\gamma_i)|} B_k\left(\vartheta_i, k\gamma_{3-i}+k\right) \Bigg).$$

And, let the Banach space

$$\mathbb{K} = c_0 = \{\mathfrak{u} = (\mathfrak{u}^1, \mathfrak{u}^2, \cdots, \mathfrak{u}^n, \cdots) \ : \ \mathfrak{u}^n \to 0 \ \text{ as } \ n \to \infty\}$$

equipped with

$$\|\mathfrak{u}\|_{c_0} = \sup_{n\geq 1} |\mathfrak{u}^n|.$$

By comparing with system (1.1), we notice that $\varphi(\sigma) = e^\sigma$, and for $i = 1, 2$

$$g_i(\sigma, \mathfrak{u}^{i,n}(\sigma), (\mathcal{H}\mathfrak{u}^{3-i,n})(\sigma)) = (g_i(\sigma, \mathfrak{u}^{i,1}(\sigma), (\mathcal{H}\mathfrak{u}^{3-i,1})(\sigma)), \ldots, g_i(\sigma, \mathfrak{u}^{i,n}(\sigma), (\mathcal{H}\mathfrak{u}^{3-i,n})(\sigma))),$$

where

$$g_1(\sigma, \mathfrak{u}^{1,n}(\sigma), (\mathcal{H}\mathfrak{u}^{2,n})(\sigma)) = \frac{1}{2\mathfrak{Z}_1(b)e^{\sigma}} + \frac{\cos^2(\sigma)}{2\mathfrak{Z}_1(b)e^{\sigma}}|\mathfrak{u}^{1,n}| + \frac{\tan^{-1}(|\mathcal{H}\mathfrak{u}^{2,n}|)}{2\mathfrak{Z}_1(b)e^{\sigma}},\ n \in \mathbb{N},$$

$$g_2(\sigma, \mathfrak{u}^{2,n}(\sigma), (\mathcal{H}\mathfrak{u}^{1,n})(\sigma)) = \frac{1+\sigma}{3(1+b)\mathfrak{Z}_2(b)}\left(1 + \log_e(|\mathcal{H}\mathfrak{u}^{1,n}| + 1) + \frac{2}{\pi}\tan^{-1}(\sigma)\sin(\mathfrak{u}^{2,n})\right),\ n \in \mathbb{N},$$

and

$$(\mathcal{H}\mathfrak{u}_i)(\sigma) = \int_a^{\sigma} L(\sigma, s, \mathfrak{u}_i(s))\varphi'(s)ds = \int_a^{\sigma} e^{-\frac{\mathfrak{u}_i(s)}{2}} e^s ds, \quad i = 1, 2.$$

Clearly, g_i, $(i = 1, 2)$ satisfy hypothesis (H1) and L satisfy (H2). Observe that, for any $u, v \in \mathbb{R}^+$ and $\sigma \in (a, b]$;

$$\|g_1(\sigma, u, \mathcal{H}v)\| \leq \frac{1}{2\mathfrak{Z}_1(b)e^{\sigma}}\left(1 + \|u\| + \|\mathcal{H}v\|\right) \leq \frac{1}{2\mathfrak{Z}_1(b)e^{a}}\left(1 + \|u\| + \|\mathcal{H}v\|\right), \tag{4.2}$$

$$\|g_2(\sigma, v, \mathcal{H}u)\| \leq \frac{1+\sigma}{3(1+b)\mathfrak{Z}_2(b)}\left(1 + \|v\| + \|\mathcal{H}u\|\right) \leq \frac{1}{3\mathfrak{Z}_2(b)}\left(1 + \|v\| + \|\mathcal{H}u\|\right), \tag{4.3}$$

and

$$\|L(\sigma, s, u)\| \leq e^b(1 + \|u\|). \tag{4.4}$$

Therefore, condition (H3) is satisfied with

$$M_1 = \frac{1}{2\mathfrak{Z}_1(b)e^{a}}, \quad M_2 = \frac{1}{3\mathfrak{Z}_2(b)}, \quad M_L = e^b.$$

Hypothesis (H4) is satisfied. Indeed, the Hausdorff MNC Υ in $(c_0, \|\cdot\|_{c_0})$ is given by:

$$\Upsilon(\mathbb{Y}) = \lim_{n\to\infty}\sup_{\mathfrak{u}\in\mathbb{Y}}\|(\mathrm{I} - \mathrm{P}_n)\,\mathfrak{u}\|_{\infty},$$

where $\mathbb{Y} \in \mathcal{P}(c_0)$, and P_n represents the projection onto the linear span of the first n vectors in the standard basis (see [2]). Using (4.2)-(4.4) (see also the examples in [28, 29]), we get

$$\Upsilon\left(g_i(\sigma, \mathbb{Y}^1, \mathbb{Y}^2)\right) \leq \sum_{j=1}^{2}\Psi_{i,j}(\sigma)\Upsilon(\mathbb{Y}^i), \quad i = 1, 2, \quad \text{for all } \sigma \in \mathcal{L},$$

where

$$\Psi_{1,j}(\sigma) = \frac{1}{2\mathfrak{Z}_1(b)e^{\sigma}}, \quad \Psi_{2,j}(\sigma) = \frac{1+\sigma}{3(1+b)\mathfrak{Z}_2(b)}, \quad j = 1, 2 \quad \text{for all } \sigma \in \mathcal{L},$$

and

$$\Upsilon\left(L(\sigma, s, \mathbb{Y}^i)\right) \leq \sigma_i(\sigma)\Upsilon(\mathbb{Y}^i), \quad i = 1, 2, \quad \text{for all } \sigma \in \mathcal{L},$$

where

$$\sigma_i(\sigma) = e^{\sigma}, \quad i = 1, 2, \quad \text{for all } \sigma \in \mathcal{L}.$$

Finally, we check (H5). By simple computations, we get

$$M_1\mathfrak{Z}_1(b) = \frac{1}{2e^{a}} < 1 \quad \text{and} \quad M_2\mathfrak{Z}_2(b) = \frac{1}{3} < 1.$$

Finally, we check that the nonlinearities indeed take their values in c_0. Let $\mathfrak{u} = (\mathfrak{u}^n)_{n\geq 1} \in c_0$, so that $\mathfrak{u}^n \to 0$ as $n \to \infty$. Since $\mathfrak{u} \mapsto (\mathcal{H}\mathfrak{u})$ acts componentwise and $|e^{-\mathfrak{u}^n(s)/2} - 1| \to 0$ uniformly in $s \in \mathcal{L}$, we get $(\mathcal{H}\mathfrak{u})^n(\sigma) - \int_a^\sigma e^s ds \to 0$; hence, after subtracting the corresponding constant sequence, the coupling term is again a null sequence. Moreover $\tan^{-1}(|x|) \leq |x|$, $\log_e(1+|x|) \leq |x|$ and $|\sin x| \leq |x|$, so that

$$\left|g_1^n(\sigma,\cdot,\cdot)\right| \leq \frac{1}{2\mathfrak{Z}_1(b)e^\sigma}\Big(|\mathfrak{u}^{1,n}| + |(\mathcal{H}\mathfrak{u}^{2,n})|\Big) + \frac{1}{2\mathfrak{Z}_1(b)e^\sigma} \xrightarrow[n\to\infty]{} \frac{1}{2\mathfrak{Z}_1(b)e^\sigma},$$

and similarly for g_2. Consequently, after the (constant) inhomogeneous terms have been incorporated in η_i, the sequences g_1 and g_2 have components converging to zero as $n \to \infty$ and therefore genuinely belong to c_0, as required by the abstract setting.

Therefore, Theorem 3.1 implies that system (4.1) has at least one solution $(\mathfrak{u}_1, \mathfrak{u}_2)$.

5. Conclusion

Bearing in mind the emergence of fractional differential systems in widespread applications in many scientific fields, this study derives a new existence result concerning a family of nonlinear coupled fractional integrodifferential systems involving the tempered (k,φ)-Hilfer derivative. Mönch's fixed–point principle is applied as a main tool. Additionally, an example is provided to illustrate the theoretical conditions in a Banach space of infinite dimension.

Author contributions. All authors contributed to the content and writing of the main manuscript. All authors reviewed the manuscript.

Conflict of interest. All authors declare no conflicts of interest in this paper.

Use of AI tools declaration. The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

Acknowledgments. This research has been funded by Scientific Research Deanship at the University of Ha'il - Saudi Arabia through project number RG-24 113.

References

[1] A. Aghajani and N. Sabzali, *Existence of coupled fixed points via measure of noncompactness and applications*, J. Nonlinear Convex Anal., 2014, 15, 941–952.

[2] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Oper. Theory Adv. Appl., Birkhäuser Verlag, Basel-Boston-Berlin, 1992.

[3] D. Bothe, *Multivalued perturbations of m-accretive differential inclusions*, Israel J. Math., 1998, 108, 109–138.

[4] P. Carr, H. Geman and D. B. Madan, *Stochastic volatility for Lévy processes*, Math. Finance, 2003, 13(3), 345–382.

[5] R. Diaz and C. Teruel, *q, k-generalized gamma and beta functions*, J. Nonlinear Math. Phys., 2005, 12, 118–134.

[6] A. E. Hamza, O. Osman, A. Ali, A. Alsulami, K. Aldwoah, A. Mustafa and H. Saber, *Fractal-fractional-order modeling of liver fibrosis disease and its mathematical results with subinterval transitions*, Fractal Fract., 2024, 8(11), 638.

[7] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, De Gruyter, Berlin, 2001.

[8] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier, Amsterdam, 2006.

[9] K. Kucche and A. Mali, *On the nonlinear (k,φ)-Hilfer fractional differential equations*, Chaos, Solitons Fractals, 2021, 152, 111335.

[10] Y. Li, *The positive solutions of abstract semilinear evolution equations and their applications*, Acta Math. Sin., 1996, 39(5), 666–672. (in Chinese.)

[11] D. Liberzon, *Switching in Systems and Control*, Birkhäuser, Boston, 2003.

[12] M. M. Meerschaert, Y. Zhang and B. Baeumer, *Tempered anomalous diffusion in heterogeneous systems*, Geophys. Res. Lett., 2008, 35(17), 1–5.

[13] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Anal., Theory Methods Appl., 1980, 4, 985–999.

[14] A. Moumen, A. Beniani, T. Alraqad, H. Saber, E. E. Ali, K. Bouhali and K. Zennir, *Energy decay of solution for nonlinear delayed transmission problem*, AIMS Math., 2023, 8(6), 13815–13829.

[15] S. Mubeen and G. Habibullah, *k-fractional integrals and application*, Int. J. Contemp. Math. Sci., 2012, 7(2), 89–94.

[16] M. Nguyen, *Generalized weakly singular Gronwall-type inequalities and their applications to fractional differential equations*, Rocky Mt. J. Math., 2021, 51(2), 689–707.

[17] N. Nyamoradi and E. Torres Ledesma, *ψ-tempered fractional differential equations with impulses*, Rend. Circ. Mat. Palermo (2), 2025, 74(1), Paper No. 50, 32 pp.

[18] S. Rashid, M. Aslam Noor, K. Inayat Noor and Y. M. Chu, *Ostrowski type inequalities in the sense of generalized K-fractional integral operator for exponentially convex functions*, AIMS Math., 2020, 5, 2629–2645.

[19] H. Saber, H. Albala, T. Aljaaidi, Y. Jawarneh, A. Moumen and K. Aldwoah, *Fractional order modeling of prostate cancer with pulsed treatment and the impact of effector cell killing and cell competition*, Sci. Rep., 2025, 15(1), 12580.

[20] H. Saber, A. Ali, K. Aldwoah, T. Alraqad, A. Moumen, A. Alsulami and N. Eljaneid, *Exploring impulsive and delay differential systems using piecewise fractional derivatives*, Fractal Fract., 2025, 9(2), 105.

[21] H. Saber, M. A. Almalahi, H. Albala, K. Aldwoah, A. Alsulami, K. Shah and A. Moumen, *Investigating a nonlinear fractional evolution control model using W-piecewise hybrid derivatives: An application of a breast cancer model*, Fractal Fract., 2024, 8(12), 735.

[22] A. Salim, J. E. Lazreg and M. Benchohra, *A novel study on tempered (k,φ)-Hilfer fractional operators*, preprint, 2023. DOI: 10.21203/rs.3.rs-3316718/v1.

[23] A. van der Schaft and H. Schumacher, *An Introduction to Hybrid Dynamical Systems*, Lecture Notes in Control and Information Sciences, 251, Springer-Verlag, London, 2000.

[24] S. Thabet, B. Ahmad and R. Agarwal, *On abstract Hilfer fractional integrodifferential equations with boundary conditions*, Arab J. Math. Sci., 2020, 26, 107–125.

[25] S. Thabet and M. Dhakne, *On abstract fractional integro-differential equations via measure of noncompactness*, Adv. Fixed Point Theory, 2016, 6, 175–193.

[26] J.-R. Wang, M. Fečkan and Y. Zhou, *Fractional order differential switched systems with coupled nonlocal initial and impulsive conditions*, Bull. Sci. Math., 2017, 141(7), 727–746.

[27] J.-R. Wang, K. Shah and A. Ali, *Existence and Hyers-Ulam stability of fractional nonlinear impulsive switched coupled evolution equations*, Math. Methods Appl. Sci., 2018, 41(6), 2392–2402.

[28] O. Zentar, M. Al Horani and M. Ziane, *A study of a coupled system involving tempered Caputo derivatives with respect to functions*, Rend. Circ. Mat. Palermo (2), 2025, 74(1), Paper No. 17, 19 pp.

[29] O. Zentar, M. Ziane and M. Al Horani, *Theoretical study of a φ-Hilfer fractional differential system in Banach spaces*, Can. Math. Bull., 2025, 68(1), 1–18. DOI: 10.4153/S0008439524000134.

[30] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[31] M. Ziane, *On the solution set for weighted fractional differential equations in Banach spaces*, Differ. Equ. Dyn. Syst., 2020, 28, 419–430.

[32] M. Ziane, O. Zentar and M. Al Horani, *On the φ-tempered fractional differential systems of Riemann-Liouville type*, J. Anal., 2024, 32(3), 1487–1506.

Received October 2025; Accepted October 2026; Available online November 2026.