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HYERS-ULAM STABILITY OF GENERAL BILINEAR FUNCTIONAL EQUATIONS: NEW RESULTS AND COMPUTATIONAL INSIGHTS*

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Abstract The goal of this paper is to investigate the Hyers-Ulam stability of the general bilinear functional equation. To achieve this goal, a fixed-point method for products of spaces is constructed. With relevant hypotheses imposed and a fixed-point theorem proven, together with mathematical deductions including induction, the effectiveness of this method for analyzing the stability of functional equations is verified. The research results show that under specific coefficient conditions, there exists a unique general bilinear mapping satisfying the stability bound of the functional equation. In addition, as corollaries, some stability results for several well-known functional equations are derived. This study establishes a new strategy for analyzing the Hyers-Ulam stability of functional equations and provides theoretical support for exploring the asymptotic behaviors of various functional equations in Banach spaces.

Keywords Fixed point theorem, bilinear functional equation, Hyers-Ulam stability.

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1. Introduction and preliminaries

In what follows $\mathbb{N}, \mathbb{R}$ and $\mathbb{K}$ denote the sets of positive integers, real numbers, and the real- or complex-number field, respectively. Moreover, X^Y stands for the set of all functions from a non-empty set X to a non-empty set Y. Let $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}, \mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Suppose X is a linear space on the field $\mathbb{F}$ and Y is a linear space on the field $\mathbb{K}$. The function $f : X \times X \to Y$ is defined *bilinear* if it satisfies

$$\begin{aligned} &f(a_1x + a_2y, z) = Af(x, z) + Bf(y, z), \\ &f(x, b_1y + b_2z) = Cf(x, y) + Df(x, z),\ x, y, z \in X, \end{aligned}$$

with some scalars $a_1, a_2, b_1, b_2 \in \mathbb{F}$ and $A, B, C, D \in \mathbb{K}$. When $a_1 = a_2 = b_1 = b_2 = 1$ and $A = B = C = D = 1$, then the function is called a bi-additive function. The definition of n-linear (n-additive) can be seen in [20].

Let $a, b, c, d \in \mathbb{F}$ and $A, B, C, D \in \mathbb{K}$ be given scalars. The equation

$$f(ax + by, cl + dz) = Af(x, l) + Bf(x, z) + Cf(y, l) + Df(y, z),\ x, y, l, z \in X, \tag{1.1}$$

is called the general bilinear functional equation.

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We now recall some background facts concerning stability problems for functional equations. The subject originates from a problem posed by Ulam [53] in 1940. Later, Hyers [24] gave the first partial affirmative answer to Ulam's question within the setting of Banach spaces, which laid the foundation for subsequent developments. Comprehensive historical surveys can be found in monographs by Hyers, Isac and Rassias [25], Jung [29], Czerwik [21], and Kannappan [33]. Aczél and Dhombres provided fundamental background for multivariable functional equations in several variables in their classic volume [3].

Hyers' original result was extended by Aoki to additive mappings with power-type perturbation bounds [5], whereas Rassias further generalised the theory for linear mappings by allowing unbounded Cauchy differences [47]. Găvruta introduced abstract control function perturbations, which unified many earlier stability criteria [22]. Wang [54] developed further generalisations of Ulam-Hyers-Rassias stability for broader families of functional equations. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 16, 25, 27–29, 34, 38, 44, 48–50, 52]). Direct methods based on iterative averaging and limiting procedures dominated early investigations; over the last two decades, fixed-point techniques have emerged as a powerful complementary toolkit. Brzdęk and co-workers systematically advanced the fixed-point approach for functional-equation stability, summarising many key advances in edited volumes [12, 13]. The survey by Ciepliński [19] offers an overview of fixed-point applications across Ulam-stability research.

Beyond classical Banach space settings, stability theory has been extended to diverse normed and probabilistic structures. Cho, Rassias and Saadati developed stability results for random normed spaces and Banach algebra frameworks in two monographs [17, 18]. Mohammadpour et al. treated general functional equations in m Banach spaces, which represent one important direction towards multi-normed abstract settings [39]. Stability questions also arise for differential equations: Murali et al. studied Ulam stability combined with integral transforms for linear second order ODEs [42]; Alinejad et al. considered inner product related functional equations within Serstnev probabilistic normed spaces [4]. Abdollahpour et al. investigated Hyers-Ulam stability for special classes of hypergeometric and Laguerre differential equations [1, 2]. Parallel lines of research address mixed-type functional equations: Lee et al. established uniqueness statements for cubic-quadratic-additive inequalities and high dimensional mixed additive-quadratic systems [36, 37]; Mortici and Jung separately analysed stability for Fibonacci-related functional equations [30, 41]; Jung also examined trigonometric type functional equations and stability on complete metric groups [31, 32]. For operator algebraic contexts, Park et al. studied additive functional equations and partial multipliers in C^*-algebras [46], complementing earlier works on approximate homomorphisms by Hyers and Rassias [26]. Sahoo and Riedel gave an accessible introductory treatment of functional equation techniques [51]. Early foundational work on approximate transformations was contributed by Bourgin [9].

The Brzdek's fixed point theorem has been shown in order to obtain the stability result for the generalized additive functional equation in Banach spaces. By using the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [11, 14, 15, 19, 23, 34, 40, 43, 45, 55]).

The research on the stability of bilinear functional equations, which combines both bilinear characteristics and functional properties, has an especially close connection with algebraic structures. In recent years, most studies have focused on special forms of equations within specific algebraic structures: In 2025, Koh [35], in Symmetry, investigated the Hyers-Ulam-Rassias stability of reciprocal-type functional equations in non-Archimedean fuzzy normed spaces by ap-

plying the direct method and Brzdek's fixed-point method. This work provides a methodological reference for the research on special forms of bilinear equations.

The literature [6] pointed out that the generalized stability of the previously studied multi-variable bi-additive functional equations in C*-ternary algebras remained unsolved. Through the construction of special control functions, this study established stability results of multi-variable bi-additive equations with respect to bi-homomorphisms, bi-derivations, and bi-isomorphisms, pushing the research on bilinear-related equations toward more complex algebraic scenarios [8].

To address the lack of stability in generalized multivariate bilinear equations, we draw on Găvruta's control function idea to construct a stability framework for n-ary bilinear equations in Banach space. We construct a family of multi-parameter control functions, derive convergence criteria for approximate bilinear mappings and their exact limits, and prove a universal existence theorem. In particular, we modify the fixed-point method by replacing the classical Banach contraction with a quasi-contraction condition, which offers greater flexibility for error estimation. The fixed-point theorem presented below will be verified on a product space, and we first introduce the following hypotheses.

(1) X is a nonempty set and $(Y, \|\cdot\|)$ is a Banach space.

(2) $f_1, \cdots, f_k, g_1, \cdots, g_k : X \to X$ and $L_1, \cdots, L_k : X \times X \to R^+$ are given mappings.

(3) $\mathcal{T} : Y^{X\times X} \to Y^{X\times X}$ is an operator satisfying the inequality

$$\begin{aligned} &\|((\mathcal{T}\xi)(x,y) - (\mathcal{T}\mu)(x,y))\| \\ &\leq \sum_{i=1}^{k} L_i(x,y)\|(\xi(f_i(x),g_i(y)) - \mu(f_i(x),g_i(y)))\|,\ \xi,\mu \in Y^X, x,y \in X. \end{aligned} \tag{1.2}$$

(4) $\triangle$ is a linear operator defined by

$$(\triangle\delta)(x,y) := \sum_{i=1}^{k} L_i(x,y)\delta(f_i(x),g_i(y)) \tag{1.3}$$

for $\delta : X \times X \to \mathbb{R}^+$ and $x, y \in X$. Clearly, $\triangle$ is monotone with respect to the pointwise ordering in $(\mathbb{R}^+)^{X\times X}$ (provided that L_i is nonnegative).

Theorem 1.1. *Let the hypotheses (1)-(4) hold and $\varepsilon : X \times X \to \mathbb{R}^+$ be a function and $\phi : X \times X \to Y$ be a mapping such that, for all $x, y \in X$,*

$$\|(\mathcal{T}\phi)(x,y) - \phi(x,y)\| \leq \varepsilon(x,y) \tag{1.4}$$

and

$$\sum_{n=0}^{\infty}(\triangle^n\varepsilon)(x,y) =: \varepsilon^*(x,y) < \infty. \tag{1.5}$$

Then, for all $(x, y) \in X \times X$, the limit

$$\lim_{n\to\infty}(\mathcal{T}^n\phi)(x,y) := \varphi(x,y), \tag{1.6}$$

exists and the mapping $\varphi : X \times X \to Y$ is a fixed point of $\mathcal{T}$ satisfying

$$\|\phi(x,y) - \varphi(x,y)\| \leq \varepsilon^*(x,y),\ \forall (x,y) \in X \times X.$$

Proof. The following inequality can be shown by induction that, for every $n \in \mathbb{N}_0$,

$$\|(\mathcal{T}^{n+1}\phi)(x,y) - (\mathcal{T}^n\phi)(x,y)\| \le (\triangle^n\varepsilon)(x,y),\ (x,y) \in X \times X. \tag{1.7}$$

Obviously, by (1.4), the following inequality can be obtained

$$\|(\mathcal{T}\phi)(x,y) - \phi(x,y)\| \le \varepsilon(x,y) = (\triangle^0\varepsilon)(x,y),\ (x,y) \in X \times X. \tag{1.8}$$

Now, fix $n \in \mathbb{N}^+$ and let us suppose that (1.7) is true. Then, using hypothesis (3), for any $(x,y) \in X \times X$, we get

$$\begin{aligned} \|(\mathcal{T}\phi)(x,y) - (\mathcal{T}\phi)(x,y)\| &\le \sum_{i=1}^{k} L_i(x,y)\|(\mathcal{T}^n\phi)(f_i(x),g_i(y)) - (\mathcal{T}\phi)(f_i(x),g_i(y))\| \\ &\le \sum_{i=1}^{k} L_i(x,y)(\triangle^n\varepsilon)(f_i(x),g_i(y)) \\ &= (\triangle^{n+1}\varepsilon)(x,y). \end{aligned} \tag{1.9}$$

Thus, the inequality (1.7) holds for any $n \in \mathbb{N}_0$.

Next, for any $n, k \in \mathbb{N}_0, k > 0$,

$$\begin{aligned} \|(\mathcal{T}^{n+k}\phi)(x,y) - (\mathcal{T}^n\phi)(x,y)\| &\le \sum_{i=0}^{k-1} \|(\mathcal{T}^{n+i}\phi)(x,y) - (\mathcal{T}^{n+i+1}\phi)(x,y)\| \\ &\le \sum_{i=n}^{n+k-1} (\triangle^i\varepsilon)(x,y) \\ &\le \varepsilon^*(x,y) \\ &< \infty,\ (x,y) \in X \times X. \end{aligned} \tag{1.10}$$

By the convergence of the series $\sum(\triangle^n\varepsilon)(x,y)$, it follows from (1.10) that $((\mathcal{T}^n\phi)(x,y))_{n\in\mathbb{N}}$ is Cauchy sequence and Y is Banach space, and so the limit $\varphi(x,y) := \lim_{n\to\infty}(\mathcal{T}^n\phi)(x,y)$ exists for all $(x,y) \in X \times X$. Taking $n = 0$ and letting $k \to \infty$ in the equality (1.10), we can find that $\mathcal{T}(\varphi) = \varphi$ in view of the inequality (1.3).

In order to prove the uniqueness of φ, suppose that $\psi \in Y^{X\times X}$ is also a fixed point of $\mathcal{T}$ with $\|\phi(x,y) - \psi(x,y)\| \le \varepsilon^*(x,y)$ for any $(x,y) \in X \times X$. Let us show that, for every $m \in \mathbb{N}$,

$$\begin{aligned} \|\varphi(x,y) - \psi(x,y)\| &= \|(\mathcal{T}^m\varphi)(x,y) - (\mathcal{T}^m\psi)(x,y)\| \\ &\le 2\sum_{i=m}^{\infty}(\triangle^i\varepsilon)(x,y),\ (x,y) \in X \times X. \end{aligned} \tag{1.11}$$

In fact, for $m = 0$,

$$\begin{aligned} \|\varphi(x,y) - \psi(x,y)\| &\le \|\phi(x,y) - \varphi(x,y)\| + \|\phi(x,y) - \psi(x,y)\| \\ &\le 2\varepsilon^*(x,y),\ (x,y) \in X \times X. \end{aligned}$$

Now, we can suppose that the inequality (1.11) is valid for some $m \in \mathbb{N}^+$. Then, for any $(x,y) \in X \times X$, we can get the following inequality by (1.3),

$$\|(\mathcal{T}^{m+1}\varphi)(x,y) - (\mathcal{T}^{m+1}\psi)(x,y)\|$$

$$
\begin{aligned}
&= \|\mathcal{T}(\mathcal{T}^m\varphi)(x,y) - \mathcal{T}(\mathcal{T}^m\psi)(x,y)\| \\
&\le \sum_{i=1}^{k} L_i(x,y)\|((\mathcal{T}^m\varphi)(f_i(x),g_i(y))) - ((\mathcal{T}^m\psi)(f_i(x),g_i(y)))\| \\
&\le 2\sum_{i=1}^{k} L_i(x,y)\left(\sum_{j=m}^{\infty}(\triangle^j\varepsilon)(f_i(x),g_i(y))\right) \\
&\le 2\sum_{j=m+1}^{\infty}(\triangle^j\varepsilon)(x,y). \qquad (1.12)
\end{aligned}
$$

If $m \to \infty$, $\varphi = \psi$ can be obtained. □

Assume that X is a linear space over the field $\mathcal{F}$, and Y is a Banach space, and let $a_i(i = 1,2,3,4), b_i(i = 1,2,3,4), c_i(i = 1,2,3,4,5) \in \mathcal{F}$, and $D_i(i = 1,2,\cdots,10) \in \mathcal{R}$ be scalars, and $D := \sum_{i=1}^{10} D_i \neq 0$. The Hyers-Ulam stability of the following general bilinear functional equation in multi-variables is examined:

$$
\begin{aligned}
&f(a_1x - a_2y, a_3z + a_4w) + f(b_1x + b_2y, b_3l - b_4k) + f(c_1x - c_2y, c_3l + c_4k - c_5m) \\
&= D_1f(x,z) + D_2f(x,w) + D_3f(y,z) + D_4f(y,w) + D_5f(x,l) \qquad (1.13) \\
&\quad + D_6f(y,l) + D_7f(x,k) + D_8f(y,k) + D_9f(x,m) + D_{10}f(y,m),
\end{aligned}
$$

where $f : X \times X \to Y$ and $x, y, l, k, m, z, w \in X$.

2. Main results

In the section, we establish the Hyers-Ulam stability of general bilinear functional equation (1.13)by means of the fixed-point method.

Theorem 2.1. *Let $H : X \times X \to \mathbb{R}_+$ be a function such that $H(mx, ny) \le |mn|H(x,y), m, n \in \mathbb{R}$. Suppose that X is a linear normed space, Y is a Banach space, and $\theta \ge 0$, let $f : X \times X \to Y$ be a mapping satisfying*

$$
\begin{aligned}
&\|f(a_1x - a_2y, a_3z + a_4w) + f(b_1x + b_2y, b_3l - b_4k) + f(c_1x - c_2y, c_3l + c_4k - c_5m) \\
&\quad - D_1f(x,z) - D_2f(x,w) - D_3f(y,z) - D_4f(y,w) - D_5f(x,l) \\
&\quad - D_6f(y,l) - D_7f(x,k) - D_8f(y,k) - D_9f(x,m) - D_{10}f(y,m)\| \qquad (2.1) \\
&\le \theta(H(x,l) + H(x,z) + H(x,w) + H(y,l) + H(y,z) + H(x,k) \\
&\quad + H(x,m) + H(y,k) + H(y,m) + H(y,w)),
\end{aligned}
$$

for all $x, y, z, l, w, k, m \in X$. If $a_i(i = 1,2,3,4)$, $b_i(i = 1,2,3,4)$, $c_i(i = 1,2,3,4,5) \in \mathbb{F}$ and $D_i(i = 1,2,\cdots,10) \in \mathbb{K}$ are not zero and different from each other. For convenience, we set $\widetilde{a_1} := a_1 - a_2$; $\widetilde{a_2} := a_3 + a_4$; $\widetilde{b_1} := b_1 + b_2$; $\widetilde{b_2} := b_3 - b_4$; $\widetilde{c_1} := c_1 - c_2$; $\widetilde{c_2} := c_3 + c_4 - c_5$; $D = D_1 + D_2 + D_3 + D_4 + \cdots + D_{10}$ and $\frac{|\widetilde{a_1}\widetilde{a_2}| + |\widetilde{b_1}\widetilde{b_2}| + |\widetilde{c_1}\widetilde{c_2}|}{|D|} < 1$. Then there exists a unique general bi-linear mapping $A : X \times X \to Y$ such that

$$
\begin{aligned}
\|f(x,y) - A(x,y)\| &\le \frac{|D|}{|D| - (|\widetilde{a_1}\widetilde{a_2}| + |\widetilde{b_1}\widetilde{b_2}| + |\widetilde{c_1}\widetilde{c_2}|)}\varepsilon(x,y) \\
&= \frac{10\theta}{|D| - (|\widetilde{a_1}\widetilde{a_2}| + |\widetilde{b_1}\widetilde{b_2}| + |\widetilde{c_1}\widetilde{c_2}|)}H(x,y),
\end{aligned}
$$

for all $x, y \in X$.

Proof. Letting $y = x$ and $l = w = m = k = z$ in the inequality (2.1), we can get

$$\left\| f(x,z) - \frac{1}{D} f(\widetilde{a_1}x, \widetilde{a_2}z) - \frac{1}{D} f(\widetilde{b_1}x, \widetilde{b_2}z) - \frac{1}{D} f(\widetilde{c_1}x, \widetilde{c_2}z) \right\| \leq \frac{10\theta}{|D|} H(x,z),\ x, z \in X. \quad (2.2)$$

Consider $J : Y^{X\times X} \to Y^{X\times X}$ and $\varepsilon : X \times X \to \mathbb{R}_+$ given as

$$J\xi(x,z) = \frac{1}{D}\xi(\widetilde{a_1}x, \widetilde{a_2}z) + \frac{1}{D}\xi(\widetilde{b_1}x, \widetilde{b_2}z) + \frac{1}{D}\xi(\widetilde{c_1}x, \widetilde{c_2}z),\ \forall x, z \in X,$$

and

$$\varepsilon(x,z) = \frac{10\theta}{|D|} H(x,z), \quad x, z \in X.$$

The inequality (2.2) becomes

$$\|Jf(x,z) - f(x,z)\| \leq \varepsilon(x,z),\ \forall x, z \in X.$$

For all $g, h \in Y^{X\times X}$ and $x, z \in X$,

$$\begin{aligned}
&\|Jg(x,z) - Jh(x,z)\| \\
&= \left\| \frac{1}{D} g(\widetilde{a_1}x, \widetilde{a_2}z) - \frac{1}{D} h(\widetilde{a_1}x, \widetilde{a_2}z) + \frac{1}{D} g(\widetilde{b_1}x, \widetilde{b_2}z) - \frac{1}{D} h(\widetilde{b_1}x, \widetilde{b_2}z) \right. \\
&\quad \left. + \frac{1}{D} g(\widetilde{c_1}x, \widetilde{c_2}z) - \frac{1}{D} h(\widetilde{c_1}x, \widetilde{c_2}z) \right\| \\
&\leq \left\| \frac{1}{D} g(\widetilde{a_1}x, \widetilde{a_2}z) - \frac{1}{D} h(\widetilde{a_1}x, \widetilde{a_2}z) \right\| + \left\| \frac{1}{D} g(\widetilde{b_1}x, \widetilde{b_2}z) - \frac{1}{D} h(\widetilde{b_1}x, \widetilde{b_2}z) \right\| \\
&\quad + \left\| \frac{1}{D} g(\widetilde{c_1}x, \widetilde{c_2}z) - \frac{1}{D} h(\widetilde{c_1}x, \widetilde{c_2}z) \right\|.
\end{aligned}$$

So J satisfies the inequality (1.2) with $f_1(x) = \widetilde{a_1}x$, $f_2(x) = \widetilde{b_1}x$, $f_3(x) = \widetilde{c_1}x$, $g_1(z) = \widetilde{a_2}z$, $g_2(z) = \widetilde{b_2}z$, $g_3(z) = \widetilde{c_2}z$ and $L_i(x,z) = \frac{1}{|D|}, (i = 1, 2, 3)$. By (1.3), the operator $\triangle : \mathbb{R}_+^{X\times X} \to \mathbb{R}_+$ is given by

$$\triangle\eta(x,z) = \frac{1}{|D|}\eta(\widetilde{a_1}x, \widetilde{a_2}z) + \frac{1}{|D|}\eta(\widetilde{b_1}x, \widetilde{b_2}z) + \frac{1}{|D|}\eta(\widetilde{c_1}x, \widetilde{c_2}z),\ \forall x, z \in X.$$

In particular,

$$\begin{aligned}
\triangle\varepsilon(x,z) &= \frac{1}{|D|}\varepsilon(\widetilde{a_1}x, \widetilde{a_2}z) + \frac{1}{|D|}\varepsilon(\widetilde{b_1}x, \widetilde{b_2}z) + \frac{1}{|D|}\varepsilon(\widetilde{c_1}x, \widetilde{c_2}z) \\
&= \frac{10\theta}{|D|^2} H(\widetilde{a_1}x, \widetilde{a_2}z) + \frac{10\theta}{|D|^2} H(\widetilde{b_1}x, \widetilde{b_2}z) + \frac{10\theta}{|D|^2} H(\widetilde{c_1}x, \widetilde{c_2}z) \\
&\leq \frac{10\theta|\widetilde{a_1}\widetilde{a_2}|}{|D|^2} H(x,z) + \frac{10\theta|\widetilde{b_1}\widetilde{b_2}|}{|D|^2} H(x,z) + \frac{10\theta|\widetilde{c_1}\widetilde{c_2}|}{|D|^2} H(x,z) \\
&= \frac{|\widetilde{a_1}\widetilde{a_2}| + |\widetilde{b_1}\widetilde{b_2}| + |\widetilde{c_1}\widetilde{c_2}|}{|D|} \frac{10\theta}{|D|} H(x,z) \\
&= \frac{|\widetilde{a_1}\widetilde{a_2}| + |\widetilde{b_1}\widetilde{b_2}| + |\widetilde{c_1}\widetilde{c_2}|}{|D|} \varepsilon(x,z).
\end{aligned}$$

Since $\triangle$ is linear, we can get

$$\triangle^n\varepsilon(x,z) \le \left\{\frac{|\widetilde{a_1}\widetilde{a_2}| + |\widetilde{b_1}\widetilde{b_2}| + |\widetilde{c_1}\widetilde{c_2}|}{|D|}\right\}^n \varepsilon(x,z),\ x,z \in X, n \in \mathbb{N}_0.$$

Since $\frac{|\widetilde{a_1}\widetilde{a_2}|+|\widetilde{b_1}\widetilde{b_2}|+|\widetilde{c_1}\widetilde{c_2}|}{|D|} < 1$, the series $\sum_{n=0}^{\infty}\triangle^n\varepsilon(x,z)$ is convergent for every $x,z \in X$ and

$$\begin{aligned}
\varepsilon^*(x,z) &= \sum_{n=0}^{\infty}\triangle^n\varepsilon(x,z)\\
&\le \sum_{n=0}^{\infty}\left(\frac{|\widetilde{a_1}\widetilde{a_2}| + |\widetilde{b_1}\widetilde{b_2}| + |\widetilde{c_1}\widetilde{c_2}|}{|D|}\right)^n \varepsilon(x,z)\\
&= \frac{1}{1-\left\{\frac{|\widetilde{a_1}\widetilde{a_2}|+|\widetilde{b_1}\widetilde{b_2}|+|\widetilde{c_1}\widetilde{c_2}|}{|D|}\right\}}\varepsilon(x,z)\\
&= \frac{|D|}{|D| - (|\widetilde{a_1}\widetilde{a_2}| + |\widetilde{b_1}\widetilde{b_2}| + |\widetilde{c_1}\widetilde{c_2}|)}\varepsilon(x,z)\\
&= \frac{10\theta}{|D| - (|\widetilde{a_1}\widetilde{a_2}| + |\widetilde{b_1}\widetilde{b_2}| + |\widetilde{c_1}\widetilde{c_2}|)}H(x,z)
\end{aligned}$$

for all $x,z \in X$.

By Theorem 1.1, there exists a mapping $A : X \times X \to Y$ such that

$$\begin{aligned}
&A(x,z) = \lim_{n\to\infty} J^n f(x,z),\\
&A(x,z) = \frac{1}{D}A(\widetilde{a_1}x, \widetilde{a_2}z) + \frac{1}{D}A(\widetilde{b_1}x, \widetilde{b_2}z) + \frac{1}{D}A(\widetilde{c_1}x, \widetilde{c_2}z),\\
&\|f(x,z) - A(x,z)\| \le \frac{10\theta}{|D| - |\widetilde{a_1}\widetilde{a_2}| + |\widetilde{b_1}\widetilde{b_2}| + |\widetilde{c_1}\widetilde{c_2}|}H(x,z)
\end{aligned}$$

for all $x,z \in X$.

Next, we prove that A satisfies the general bi-linear functional equation. We can get $f(x,0) = f(0,z) = f(0,0) = 0$ from (2.1). Letting $l = k = m = 0$ in (2.1), we have

$$\begin{aligned}
&\|f(a_1x - a_2y, a_3z + a_4w) - D_1f(x,z) - D_2f(x,w) - D_3f(y,z) - D_4f(y,w)\|\\
&\le \theta\left(H(x,z) + H(x,w) + H(y,z) + H(y,w)\right)
\end{aligned}$$

for all $x,y,z,w \in X$. Then, from the above inequality, we get

$$\begin{aligned}
&\|Jf(a_1x - a_2y, a_3z + a_4w) - D_1Jf(x,z) - D_2Jf(x,w) - D_3Jf(y,z) - D_4Jf(y,w)\|\\
&\le \frac{1}{|D|}\|f(a_1\widetilde{a_1}x - a_2\widetilde{a_1}y, a_3\widetilde{a_2}z + a_4\widetilde{a_2}w) - D_1f(\widetilde{a_1}x, \widetilde{a_2}z) - D_2f(\widetilde{a_1}x, \widetilde{a_2}w)\\
&\quad -D_3f(\widetilde{a_1}y, \widetilde{a_2}z) - D_4f(\widetilde{a_1}y, \widetilde{a_2}w)\|\\
&\quad + \frac{1}{|D|}\Big\|f(a_1\widetilde{b_1}x - a_2\widetilde{b_1}y, a_3\widetilde{b_2}z + a_4\widetilde{b_2}w) - D_1f(\widetilde{b_1}x, \widetilde{b_2}z) - D_2f(\widetilde{b_1}x, \widetilde{b_2}w)\\
&\quad -D_3f(\widetilde{b_1}y, \widetilde{b_2}z) - D_4f(\widetilde{b_1}y, \widetilde{b_2}w)\Big\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|D|}\|f(a_1\widetilde{c_1}x-a_2\widetilde{c_1}y,a_3\widetilde{c_2}z+a_4\widetilde{c_2}w)-D_1f(\widetilde{c_1}x,\widetilde{c_2}z)-D_2f(\widetilde{c_1}x,\widetilde{c_2}w)\\
&-D_3f(\widetilde{c_1}y,\widetilde{c_2}z)-D_4f(\widetilde{c_1}y,\widetilde{c_2}w)\|\\
\le&\frac{\theta}{|D|}\left(H(\widetilde{a_1}x,\widetilde{a_2}z)+H(\widetilde{a_1}x,\widetilde{a_2}w)+H(\widetilde{a_1}y,\widetilde{a_2}z)+H(\widetilde{a_1}y,\widetilde{a_2}w)\right.\\
&+H(\widetilde{b_1}x,\widetilde{b_2}z)+H(\widetilde{b_1}x,\widetilde{b_2}w)+H(\widetilde{b_1}y,\widetilde{b_2}z)+H(\widetilde{b_1}y,\widetilde{b_2}w)\\
&\left.+H(\widetilde{c_1}x,\widetilde{c_2}z)+H(\widetilde{c_1}x,\widetilde{c_2}w)+H(\widetilde{c_1}y,\widetilde{c_2}z)+H(\widetilde{c_1}y,\widetilde{c_2}w)\right)\\
\le&\frac{4\theta(|\widetilde{a_1}\widetilde{a_2}|+|\widetilde{b_1}\widetilde{b_2}|+|\widetilde{c_1}\widetilde{c_2}|)}{|D|}(H(x,z)+H(x,w)+H(y,z)+H(y,w))
\end{aligned}
$$

for all $x,y,z,w\in X$. Hence

$$
\begin{aligned}
&\|J^nf(a_1x-a_2y,a_3z+a_4w)-D_1J^nf(x,z)-D_2J^nf(x,w)-D_3J^nf(y,z)-D_4J^nf(y,w)\|\\
\le&\frac{4\theta(|\widetilde{a_1}\widetilde{a_2}|+|\widetilde{b_1}\widetilde{b_2}|+|\widetilde{c_1}\widetilde{c_2}|)^n}{|D|^n}(H(x,z)+H(x,w)+H(y,z)+H(y,w))
\end{aligned}
$$

for all $n\in\mathbb{N}_0$ and $x,y,z,w\in X$. Letting $n\to\infty$, we obtain

$$A(a_1x-a_2y,a_3z+a_4w)=D_1A(x,z)+D_2A(x,w)+D_3A(y,z)+D_4A(y,w),$$

for all $x,y,z,w\in X$. This completes the proof. □

3. Conclusion

We established a new strategy for the study Hyers-Ulam stability of the functional equation given in Abstract. As a consequence, we applied the obtained results to investigate several asymptotic behaviors of functions equation in Banach spaces.

Declarations

Availablity of data and materials. Not applicable.

Conflict of interest. The authors declare that they have no competing interests.

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Authors' contributions. G. Lyu: Methodology (developed the core stability analysis method), Formal Analysis (led mathematical derivations), Writing - Original Draft (drafted main content); M. Fang: Supervision (oversaw research design), Validation (verified result accuracy), Writing - Review and Editing (polished structure/expression); Y. Jin: Organized normed space investigation; Q. Liu: Writing - Editing (refined technical terminology). All authors read and approved the final manuscript.

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