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A NEW COMPACT CONDITION WITH APPLICATIONS TO NON-PERIODIC DISCRETE NONLINEAR SCHRÖDINGER EQUATIONS WITH LOCAL SUPER-QUADRATIC CONDITIONS*

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Abstract We consider the following discrete nonlinear Schrödinger equation

$$\begin{cases} -\Delta u(n) + V(n)u(n) - \omega u(n) = f(n, u(n)), & n \in \mathbb{Z} \\ \lim_{|n|\to\infty} u(n) = 0, \end{cases}$$

with non-periodic potentials. In this paper, for the first time, we present a new compact condition weaker than the classical coercive condition on V, and prove the existence and multiplicity of nontrivial homoclinic solutions under the local super-quadratic condition. Even for the global super-quadratic case, our results extend and improve some existing ones in the literature. Moreover, the new compact condition can also be suitably modified to study the continuous nonlinear Schrödinger equations.

Keywords New compact condition, discrete nonlinear Schrödinger equation, non-periodic potential, local super-quadratic condition.

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1. Introduction and main results

The discrete nonlinear Schrödinger (DNLS) equations are one of the most important inherently discrete models, having a crucial role in describing a great variety of phenomena such as nonlinear optics [8], biomolecular chain [12], Bose-Einstein condensates [17] and biological model [24,35]. Some reviews on DNLS equations can be found in [10], especially both theoretical and numerical simulation results can be found in [9, 11].

Due to the importance of DNLS equations, we consider standing waves of the discrete nonlinear Schrödinger equation

$$i\dot{\psi_n} = -\Delta\psi_n + V(n)\psi_n - f(n, \psi_n), \quad n \in \mathbb{Z}, \tag{1.1}$$

where Δ is the discrete Laplacian in one spatial dimension given by $\Delta\psi_n := \psi_{n+1} + \psi_{n-1} - 2\psi_n$, and the discrete potential $\{V(n)\}$ is a sequence of real numbers. By the definition of standing waves, we assume $\psi_n = u(n)e^{-i\omega t}$ and $\lim\limits_{|n|\to\infty} \psi(n) = 0$, where $\{u(n)\}$ is a real valued sequence

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and $\omega \in \mathbb{R}$ is the temporal frequency. Furthermore, the nonlinearity f is assumed to be gauge invariant, i.e.,

$$f(n, e^{i\theta}s) = e^{i\theta}f(n, s), \qquad \forall\, \theta \in \mathbb{R}, \quad (n, s) \in \mathbb{Z} \times \mathbb{R}.$$

Then (1.1) becomes

$$\begin{cases} -\Delta u(n) + V(n)u(n) - \omega u(n) = f(n, u(n)), & n \in \mathbb{Z}, \\ \lim\limits_{|n|\to\infty} u(n) = 0. \end{cases} \tag{1.2}$$

Thus, the problem of the existence of standing waves of (1.1) has been reduced to the existence of solutions of (1.2) in the space $l^2(\mathbb{Z})$ of two-sided infinite sequences. We are interested in looking for nontrivial homoclinic solutions, i.e. solutions decaying to zero at infinity that are not equal to 0 identically.

In recent years, when the nonlinearity f satisfies the following basic assumption:

(f_0) For each $n \in \mathbb{Z}$, $f(n, s)$ is continuous in $s \in \mathbb{R}$ and there exist $C_0 > 0$ and $p \in (2, \infty)$ such that

$$|f(n, s)| \le C_0(1 + |s|^{p-1}), \qquad \forall\ (n, s) \in (\mathbb{Z}, \mathbb{R}),$$

there have been many works concentrating on the periodic case (V and f are periodic in n) of (1.2) with the super-quadratic or asymptotically quadratic nonlinearity, such as [3, 5, 13, 14, 20, 21, 25, 27, 31, 38, 39]. The classical condition on the existence of nontrivial solutions for (1.2) with super-quadratic nonlinearity is (AR) due to Ambrosetti and Rabinowitz [2];

(AR) there exists $\mu > 2$ such that $0 < \mu F(n, s) \le f(n, s)s$ for $n \in \mathbb{Z}$ and $s \ne 0$, where $F(n, s) := \int_0^s f(n, t)dt$, which was proposed in the DNLS equations by Pankov in [20]. Subsequently, Mai and Zhou [19] introduced a more general super-quadratic condition inspired by Liu and Wang [16];

(SQ) $\lim\limits_{|s|\to\infty} \frac{F(n,s)}{s^2} = \infty$ uniformly for $n \in \mathbb{Z}$.

After that, condition (SQ) has been extensively used in the literature. We emphasize that these global super-quadratic conditions such as (AR) and (SQ) are essential in verifying the linking geometry and showing the boundedness of Cerami sequences for the energy functional associated with (1.2). Recently, Tang [28] introduced a local version of super-quadratic condition and studied nontrivial solutions for continuous nonlinear Schrödinger equations with periodic or coercive potential. Motivated by [28], Lin and Yu [14] obtained the existence of ground state solution and infinitely many geometrically distinct solutions for (1.2) under the following condition:

(f_1) There exist two integers n_* and n^* with $n_* \le n^*$ such that

$$\lim_{|s|\to\infty} \frac{F(n, s)}{s^2} = \infty \quad \text{for } n \in \Lambda := \{n_* - 1, \dots, n^* + 1\}.$$

Condition (f_1) is called local super-quadratic condition which allows the nonlinearity to be super-quadratic at some domains and asymptotically quadratic at other domains. Obviously, it weakens the usual global super-quadratic condition (SQ).

In contrast to the periodic case, there are fewer results for the non-periodic case (V and f are all non-periodic in n) of (1.2). As we know, periodicity plays an essential role in recovering the compactness due to the fact that (1.2) is defined in $\mathbb{Z}$. But non-periodic equations are quite

different from the ones described in periodic case. In addition, to overcome the difficulties caused by the unboundedness of $\mathbb{Z}$, Ma and Zhang [18,33] introduced the following condition

(V) $\lim_{|n|\to\infty} V(n) = +\infty$.

Condition (V) is crucial in showing the compactness of Palais-Smale or Cerami sequences. So (V) appeared in most of studies for non-periodic case of (1.2), see [4,6,32–34,36,37]. Recently, the authors [7] obtained ground state solution for (1.2) under the more general condition;

(V') $\lim_{|n|\to\infty} |V(n)| = +\infty$, which also puts severe restrictions on V as $|x| \to \infty$. It is an interesting question to replace (V) with different types of compact conditions. To the best knowledge of us, there seems to be very few results in this direction. In the present paper, combining the potential V and the nonlinearity f, we introduce the following condition;

(VF) for any two positive constants $a < b$,

$$\lim_{|n|\to\infty} \frac{f(n,s)}{(V(n)-\omega)s} = 0 \quad \text{uniformly for } |s| \in [a,b]. \tag{1.3}$$

It is obvious that (f_0) together with (V) implies (VF). The converse is not necessarily true. When the nonlinearity f satisfies (f_0) and (f_1), we can prove (VF) can restore the compactness of Cerami sequences (see Lemma 3.4). On this basis, we further investigate the existence of nontrivial solution for (1.2). Moreover, if f is odd in s and satisfies the stronger local super-quadratic condition:

(f_2) There exists an infinite subset $\Theta \subset \mathbb{Z}$ such that

$$\lim_{|s|\to\infty} \frac{F(n,s)}{s^2} = \infty \quad \text{for } n \in \Theta,$$

and the set Θ contains a pair of consecutive integers, we prove infinitely many high energy solutions for (1.2). It is worth pointing out that the absence of (V) together with local super-quadratic nonlinearity f brings several new difficulties in our analysis, which can be summarized as follows:

1) As is known to all, (V) ensures a compact embedding from a weighted subspace of l^2 into l^p $(p \geq 2)$, which is essential to obtain the boundedness and compactness of the Cerami sequences. But the weaker condition (VF) does not imply this nice property, see Remark 1.2. We need to make subtle analysis to overcome this difficulty.

2) Let $E = E^+ \oplus E^- \oplus E^0$ be a Hilbert space with norm $\|\cdot\|$ defined by in Section 2, and let $J \in C^1(E,\mathbb{R})$ be the energy functional associated with (1.2) defined by (2.4). To prove the existence of nontrivial homoclinic solution, we must overcome several difficulties in verifying the linking geometry and showing the boundedness of Cerami sequences for J. Specifically speaking, we need to find a suitable $e \in E^+$ and $r_0 > 0$ such that $\sup J(\partial Q) \leq 0$, where

$$Q = \big\{\omega + se \,:\, \omega \in E^- \oplus E^0,\ s \geq 0,\ \|\omega + se\| \leq r_0\big\}.$$

To this end, it seems necessary to show that

$$\lim_{\|\omega+se\|\to\infty} \sum_{n\in\Lambda} \frac{F(n,\omega(n)+se(n))}{\|\omega+se\|^2} = \infty. \tag{1.4}$$

For any given $e \in E^+$, since $\omega(n) + se(n) \neq 0$ on $\mathbb{Z}$ for any $s > 0$ and $\omega \in E^- \oplus E^0$, it follows from (SQ) and Fatou's lemma that holds if $\Lambda = \mathbb{Z}$. However, if (SQ) is replaced by (f_1), it is

very difficult to prove (1.4). The reason is that we cannot determine whether $\omega(n)+se(n)\neq 0$ on Λ for any $s>0$ and $\omega\in E^-\oplus E^0$. Second, to prove the boundedness of Cerami sequence $\{u_k\}\subset E$, one usually distinguishes two cases: $u_k/\|u_k\|\rightharpoonup 0$ and $u_k/\|u_k\|\rightharpoonup v|_{\mathbb{Z}}\neq 0$. For the second case, it is also difficult to show $v(n)\neq 0$ on Λ, which causes the difficulty in proving

$$\lim_{k\to\infty}\sum_{n\in\Lambda}\frac{F(n,u_k(n))}{\|u_k\|^2}=\infty. \tag{1.5}$$

For the proof of (1.5), we borrow some ideas from [14,28]. However, without periodic and coercive assumption on V, the arguments in [14,28] cannot be directly applied to our non-periodic case. We shall find new tricks to demonstrate the linking geometry and the boundedness of Cerami sequences for J.

3) Since the local super-quadratic condition (f_2) prevents us from using some classical multiple critical points theorems such as the Symmetry Mountain Pass theorem and Fountain theorem, see Remark 4.1. We employ a multiple critical points theorem of indefinite type introduced in [1, 2], and obtain infinitely many high energy solutions with the aid of deformation lemma and Krasnoselskii genus.

To state our results, we present the following assumptions on V and f.

(V_0) $V_\infty:=\liminf_{|n|\to\infty}V(n)>\omega$;

(V_1) there exists $n_0\in\Lambda_0:=\{n_*,\dots,n^*\}$ such that $V(n_0)-\omega+2>0$, where n_* and n^* are given in (f_1);

(f_3) $f(n,s)=o(|s|)$ as $|s|\to 0$ uniformly in $n\in\mathbb{Z}$ and $F(n,s)\geq 0$ for all $(n,s)\in(\mathbb{Z},\mathbb{R})$;

(f_4) $\mathcal{F}(n,s):=\frac{1}{2}f(n,s)s-F(n,s)\geq 0$, and there exist $C_1>0$, $R_0>0$ and $\sigma\in(0,1)$ such that for some $\nu\in(1,2/(1-\sigma)]$

$$\left[\frac{|f(n,s)|}{|s|^\sigma}\right]^\nu\leq C_1\mathcal{F}(n,s)\quad\text{for } |s|\geq R_0;$$

(f_5) for any $M>0$, the family of functions $\{g(n,s)\}_{n\in\mathbb{Z}}$ is equicontinuous on $|s|\leq M$, where

$$g(n,s):=\begin{cases}\dfrac{f(n,s)}{s}, & s\neq 0,\\ 0, & s=0.\end{cases}$$

Remark 1.1. In our proof, (f_5) is only used to show the following equality

$$J'(u_n)=J'(u_0)+J'(u_n-u_0)+o(1) \tag{1.6}$$

from the condition $u_n\rightharpoonup u_0$, where J is the energy functional associated with (1.2). In the case of $f(n,s)=a(n)h(s)$ with $a\in l^\infty(\mathbb{Z})$, it is standard to prove (1.6) under the following assumption

(h_0) $h\in C^1(\mathbb{R},\mathbb{R})$, $h(s)=o(|s|)$ as $s\to 0$ and there exist $C_0>0$ and $p\in(2,\infty)$ such that

$$|h(s)|\leq C_0(1+|s|^{p-1}),\qquad s\in\mathbb{R}.$$

However, it is more difficult to show (1.6) with a general nonlinearity f under conditions (f_0), (f_3) and (f_5).

Our main results are the following two theorems.

Theorem 1.1. *Assume that (V_0), (V_1), (VF), (f_0), (f_1), (f_3), (f_4) and (f_5) are satisfied. Then problem (1.2) has a nontrivial homoclinic solution.*

Theorem 1.2. *Assume that (V_0), (VF), (f_0), (f_2), (f_3), (f_4) and (f_5) are satisfied. Furthermore, if $f(n,-s)=-f(n,s)$ for all $(n,s)\in\mathbb{Z}\times\mathbb{R}$, then problem (1.2) possesses infinitely many high energy solutions.*

As far as we know, this is the first time the existence and multiplicity of homoclinic solutions of (1.2) with (VF) and local super-quadratic conditions have been discussed. Even for the global super-quadratic case, our work improves the related results in [4, 32–34, 37]. Moreover, the condition (VF) can be modified suitably to study continuous nonlinear Schrödinger equation. Next we present two examples to illustrate our results.

Example 1.1. In (1.2), $F(n,s)=a(n)\ln(1+s^2)s^2$, where $a\in l^\infty(\mathbb{Z})$ and $a(n)\geq 0$ for all $n\in\mathbb{Z}$. Assume that there exists an integer $n_0\in\mathbb{Z}$ such that $a(n)>0$, $n=n_0-1,\ n_0,\ n_0+1$. Suppose that $V(n_0)>\omega-2$, $\liminf_{|n|\to\infty}V(n)>\omega$ and

$$\lim_{|n|\to\infty}\frac{a(n)}{V(n)-\omega}=0.$$

Then

$$f(n,s)=a(n)\Big[2s\ln(1+s^2)+\frac{2s^3}{1+s^2}\Big],$$
$$\mathcal{F}(n,s)=\frac{a(n)s^4}{1+s^2}\geq 0.$$

It is not difficult to see that V and f satisfy all conditions in Theorem 1.1, but none of (AR), (SQ) and (V).

Example 1.2. Let Θ be an infinite subset in $\mathbb{Z}$ which contains a pair of consecutive integers, let Π be a nonempty finite set in $\mathbb{Z}$ and $\Theta\cap\Pi=\emptyset$. In (1.2), $F(n,s)=\alpha(n)|s|^{\beta(n)+2}\Big[1-\frac{1}{\ln(e+s^2)}\Big]$, where $\alpha\in l^\infty(\mathbb{Z})$, $\alpha(n)\geq 0$ for $n\in\mathbb{Z}$, $\alpha(n)>0$ for $n\in\Theta\cup\Pi$, $\lim_{|n|\to\infty}\alpha(2n)=0$, $\beta\in l^\infty(\mathbb{Z})$, $\beta(n)\geq 0$ for $n\in\mathbb{Z}$, $\beta(n)>0$ for $n\in\Theta$ and $\beta(n)=0$ for $n\in\Pi$.

$$V(n)=\begin{cases}|n|-4, & n \text{ is odd},\\ \omega-\sin n+2, & n \text{ is even}.\end{cases}\tag{1.7}$$

Then

$$f(n,s)=\alpha(n)\left\{\big(2+\beta(n)\big)|s|^{\beta(n)}s\Big[1-\frac{1}{\ln(e+s^2)}\Big]+\frac{2|s|^{\beta(n)+2}s}{(e+s^2)[\ln(e+s^2)]^2}\right\},$$
$$\mathcal{F}(n,s)=\alpha(n)\left\{\frac{\beta(n)}{2}|s|^{\beta(n)}s\Big[1-\frac{1}{\ln(e+s^2)}\Big]+\frac{|s|^{\beta(n)+4}}{(e+s^2)[\ln(e+s^2)]^2}\right\}\geq 0.$$

It is easy to see that V and f satisfy all assumptions in Theorem 1.2. Furthermore, $f(n,s)$ is super-quadratic when $n\in\Theta$ and asymptotically quadratic for $n\in\Pi$.

Remark 1.2. In Example 1.2, V and f satisfy assumption (VF). Define $\{e_i\}_{i=1}^{\infty}$ as follows:

$$e_i(n) = \begin{cases} 1, & n = 2i, \\ 0, & n \neq 2i. \end{cases}$$

Obviously, $\{e_i\}$ is bounded in E, where E is defined by (2.2). However, $\{e_i\}$ is not a Cauchy sequence in $l^q(\mathbb{Z})$ with $q \geq 2$, which implies that (VF) cannot ensure a compact embedding from E into $l^q(\mathbb{Z})$ with $q \geq 2$.

The remainder of this paper is organized as follows. In Section 2, we present some useful preliminaries. The proofs of Theorems 1.1 and 1.2 are given in Sections 3 and 4, respectively.

Throughout this paper, we denote by C_1, C_2, ... different constants in different places.

2. Preliminaries

As usual, for $1 \leq q < \infty$, set

$$l^q(\mathbb{Z}) = \left\{ \{u(n)\}_{n\in\mathbb{Z}} : \ u(n) \in \mathbb{R}, \ \ \forall\, n \in \mathbb{Z}, \ \ \sum_{n\in\mathbb{Z}} |u(n)|^q < +\infty \right\},$$

$$l^\infty(\mathbb{Z}) = \left\{ \{u(n)\}_{n\in\mathbb{Z}} : \ u(n) \in \mathbb{R}, \ \ \forall\, n \in \mathbb{Z}, \ \ \sup_{n\in\mathbb{Z}} |u(n)| < +\infty \right\},$$

and their norms are defined by

$$\|u\|_q = \Big(\sum_{n\in\mathbb{Z}} |u(n)|^q \Big)^{1/q}, \quad u \in l^q(\mathbb{Z}); \qquad \|u\|_\infty = \sup_{n\in\mathbb{Z}} |u(n)|, \quad u \in l^\infty(\mathbb{Z}).$$

Moreover, we have the embedding between such spaces:

$$l^{q_1}(\mathbb{Z}) \subset l^{q_2}(\mathbb{Z}) \ \text{ and } \ \|u\|_{q_2} \leq \|u\|_{q_1}, \ \ 1 \leq q_1 \leq q_2 \leq \infty. \tag{2.1}$$

Under assumption (V_0), we can find $m_0 > 0$ such that $\overline{V}(n) := V(n) + m_0 - \omega \geq 1$ for all $n \in \mathbb{Z}$. Then

$$E := \left\{ u \in l^2(\mathbb{Z}) : \ \sum_{n\in\mathbb{Z}} \overline{V}(n)|u(n)|^2 < \infty \right\} \tag{2.2}$$

is a Hilbert space with inner product and norm

$$(u, v)_0 := \big((-\Delta + \overline{V})u, v\big)_2, \qquad \|u\|_0 := \sqrt{(u, u)_0}, \quad \forall\, u,\, v \in E, \tag{2.3}$$

where $(\cdot,\cdot)_2$ is the inner product in $l^2(\mathbb{Z})$. Let $\mathcal{A} := -\Delta + V - \omega$. The energy functional $J : E \to \mathbb{R}$ associated with (1.2) is given by

$$J(u) = \frac{1}{2}(\mathcal{A}u, u)_2 - \sum_{n\in\mathbb{Z}} F(n, u(n)), \qquad \forall\, u \in E. \tag{2.4}$$

By (V_0) and (f_0), standard arguments show that $J \in C^1(E, \mathbb{R})$ and its derivative is given by

$$\langle J'(u), v\rangle = (\mathcal{A}u, v)_2 - \sum_{n\in\mathbb{Z}} f(n, u(n))v(n), \qquad \forall\, u,\, v \in E, \tag{2.5}$$

which implies that (1.2) is the corresponding Euler-Lagrange equation for J. Therefore, to seek nontrivial solutions of (1.2), we only need to find nonzero critical points of J in E.

It is well known that $\mathcal{A}$ is a self-adjoint operator on $l^2(\mathbb{Z})$. Next we denote by $\sigma(\mathcal{A})$ the spectrum of $\mathcal{A}$, by $\sigma_{\rm p}(\mathcal{A})$ its point spectrum and by $\sigma_{\rm ess}(\mathcal{A})$ its essential spectrum. Under assumption (V_0), $\sigma_{\rm ess}(\mathcal{A}) \subset [V_\infty - \omega, +\infty)$ due to [29]. Then we have $\sigma(\mathcal{A}) \cap (-\infty, 0] = \sigma_{\rm p}(\mathcal{A}) \cap (-\infty, 0]$, which consists of at most finite isolated eigenvalues of finite multiplicity. Next we consider the following three possible cases.

Case 1. $\sigma_{\rm p}(\mathcal{A}) \cap (-\infty, 0) \neq \emptyset$ and $0 \in \sigma_{\rm p}(\mathcal{A})$. Let

$$-\infty < \lambda_1 \leq \ldots \leq \lambda_k < 0 = \lambda_{k+1} = \ldots \lambda_{k+l}$$

denote the nonpositive eigenvalues repeated according to multiplicity. Denote by e_i the orthogonal eigenfunction corresponding to λ_i for $i \in \{1, 2, \ldots, k+l\}$ in E. Setting $E^- = \text{span}\{e_1, \ldots, e_k\}$, $E^0 = \text{span}\{e_{k+1}, \ldots, e_{k+l}\}$ and $E^+ = (E^- \oplus E^0)^\perp$, we have the orthogonal decomposition

$$E = E^+ \oplus E^- \oplus E^0, \tag{2.6}$$

where $\dim(E^- \oplus E^0) < \infty$. Obviously, the quadratic from $(\mathcal{A}u, u)_2$ is positive on E^+, negative on E^- and vanishes on E^0.

Case 2. $\sigma_{\rm p}(\mathcal{A}) \cap (-\infty, 0) = \emptyset$ and $0 \in \sigma_{\rm p}(\mathcal{A})$. Let $E^- = \{0\}$, $E^0 = \ker(\mathcal{A})$ and $E^+ = (E^- \oplus E^0)^\perp$, we have $E = E^+ \oplus E^- \oplus E^0$.

Case 3. $\sigma_{\rm p}(\mathcal{A}) \cap (-\infty, 0] = \emptyset$. Setting $E^- = \{0\}$ and $E^0 = \{0\}$, we have $E = E^+ \oplus E^- \oplus E^0$, where $E^+ = E$.

Obviously, Case 1 is the indefinite case, which is more complicated than Case 2 and Case 3. So we only give the detailed proof for Case 1. It is not difficult to present similar but easier proofs for Case 2 and Case 3. In Case 1, we introduce the following inner product and the corresponding norm on E by

$$(u, v) = (\mathcal{A}u^+, v^+)_2 - (\mathcal{A}u^-, v^-)_2 + (u^0, v^0)_2, \qquad \|u\| := \sqrt{(u, u)} \tag{2.7}$$

for any $u, v \in E = E^+ \oplus E^- \oplus E^0$, $u = u^+ + u^- + u^0$ and $v = v^+ + v^- + v^0$. Clearly, the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_0$ defined by (2.3). Moreover, the decomposition $E = E^+ \oplus E^- \oplus E^0$ is orthogonal with respect to both inner products $(\cdot, \cdot)$ and $(\cdot, \cdot)_2$. Therefore,

$$(\mathcal{A}u, u)_2 = \|u^+\|^2 - \|u^-\|^2 \tag{2.8}$$

and J can be written as

$$J(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \sum_{n \in \mathbb{Z}} F(n, u(n)) \tag{2.9}$$

for each $u = u^+ + u^- + u^0 \in E$. In addition, we also have

$$\langle J'(u), v\rangle = (u^+ - u^-, v) - \sum_{n \in \mathbb{Z}} f(n, u(n))v(n) \tag{2.10}$$

for $u = u^+ + u^- + u^0 \in E$ and $v = v^+ + v^- + v^0 \in E$.

Next we state the following result in [1, 23] which will be used in our proof of Theorem 1.1.

Lemma 2.1. *Let $X = Y \oplus Z$ be a real Hilbert space with $\dim Y < \infty$, and let $\varphi \in C^1(X, \mathbb{R})$. Suppose that the following assumption is satisfied:*

(i) There exist $r > \rho > 0$ and $e \in Z$ with $\|e\| = 1$ such that

$$\kappa := \inf \varphi(S_\rho^+) > \sup \varphi(\partial Q),$$

where

$$S_\rho^+ = \big\{u \in Z : \|u\| = \rho\big\}, \qquad Q = \big\{v + se : v \in Y,\ \|v + se\| \le r,\ s \ge 0\big\}.$$

Then there exists a constant $c \in [\kappa, \sup \varphi(Q)]$ and a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \qquad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0.$$

To overcome the difficulty caused by the local super-quadratic condition, we need the following discrete unique continuation theorem due to [14].

Lemma 2.2. *Let $\{v(n)\}$ be a real-valued sequence. Suppose that $\{u(n)\}$ satisfies $-\Delta u(n) + v(n)u(n) = 0$, $n \in \mathbb{Z}$. If $u(n_0) = u(n_0 + 1) = 0$ for some $n_0 \in \mathbb{Z}$, then $u(n) \equiv 0$ for $n \in \mathbb{Z}$.*

3. Existence of a solution

In this section, we give the proof of Theorem 1.1. First we verify the linking structure of the energy functional J in Lemma 2.1.

Lemma 3.1. *Suppose that (f_0) and (f_3) are satisfied. Then there exists $\rho > 0$ such that*

$$\kappa := \inf \big\{J(u) : u \in E^+,\ \|u\| = \rho\big\} > 0. \tag{3.1}$$

Proof. *For any $u \in E$, by (2.1) and the fact that the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_0$, there exists $\tau > 0$ such that*

$$\|u\|_p \le \|u\|_2 \le \tau\|u\|. \tag{3.2}$$

By virtue of (f_0) and (f_3), for any $\varepsilon > 0$, there exist a positive constant C_ε such that

$$|f(n, s)| \le \varepsilon|s| + C_\varepsilon|s|^{p-1}, \qquad \forall\ (n, s) \in (\mathbb{Z}, \mathbb{R}). \tag{3.3}$$

For any $u \in E^+$, by (3.2) and (3.3), we obtain

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} F(n, u(n)) \\ &\ge \frac{1}{2}\|u\|^2 - \varepsilon\|u\|_2^2 - C_\varepsilon\|u\|_p^p \\ &\ge \Big(\frac{1}{2} - \varepsilon\tau^2 - C_\varepsilon\tau^p\|u\|^{p-2}\Big)\|u\|^2. \end{aligned}$$

As a consequence, we know that

$$J(u) \ge \Big(\frac{1}{4} - C_\varepsilon\tau^p\|u\|^{p-2}\Big)\|u\|^2$$

holds true for $\varepsilon > 0$ small enough. Then we can find $\rho > 0$ small enough such that (3.1) holds. □

Define

$$e(n) = \begin{cases} 1, & n = n_0, \\ 0, & n \neq n_0, \end{cases} \tag{3.4}$$

where n_0 is given by (V_1). In view of (V_1) and (2.8), we obtain

$$\|e^+\|^2 - \|e^-\|^2 = (\mathcal{A}e, e)_2 = \sum_{n\in\mathbb{Z}} \Big[|\nabla e(n)|^2 + (V(n) - \omega)|e(n)|^2\Big] > 0,$$

where ∇ is the forward difference operator defined by $\nabla u(n) = u(n+1) - u(n)$. So we find an element $e \in E$ with $e^+ \neq 0$.

Lemma 3.2. *Suppose that (V_0), (V_1), (f_0), (f_1) and (f_3) are satisfied. Then* $\sup J(E^- \oplus E^0 \oplus \mathbb{R}^+e^+) < \infty$ *and there exists $R_e > 0$ such that*

$$J(u) \leq 0, \quad u \in E^- \oplus E^0 \oplus \mathbb{R}^+e^+, \quad \|u\| \geq R_e. \tag{3.5}$$

Proof. Arguing indirectly, assume that for some sequence $\{u_k + s_ke^+\} \subset E^- \oplus E^0 \oplus \mathbb{R}^+e^+$ with $\|u_k + s_ke^+\| \to \infty$ as $k \to \infty$ such that $J(u_k + s_ke^+) > 0$ for all $k \in \mathbb{N}$. Set

$$v_k = (u_k + s_ke^+)/\|u_k + s_ke^+\| = v_k^- + v_k^0 + t_ke^+. \tag{3.6}$$

Then $\|v_k^- + v_k^0 + t_ke^+\| = 1$. Since $\dim(E^- \oplus E^0) < \infty$, passing to a subsequence if necessary, we assume that $t_k \to t_0$, $v_k^- \to v^-$ and $v_k^0 \to v^0$ in E. Hence, it follows from (2.9) and (3.6) that

$$0 < \frac{J(u_k + s_ke^+)}{\|u_k + s_ke^+\|^2} = \frac{t_k^2}{2}\|e^+\|^2 - \frac{1}{2}\|v_k^-\|^2 - \sum_{n\in\mathbb{Z}} \frac{F(n, u_k(n) + s_ke^+(n))}{\|u_k + s_ke^+\|^2}. \tag{3.7}$$

There are three possible cases.

Case 1. $t_0 = 0$ and $v^0 = 0$. In view of (f_3) and (3.7), we have

$$0 \leq \frac{1}{2}\|v_k^-\|^2 + \sum_{n\in\mathbb{Z}} \frac{F(n, u_k(n) + s_ke^+(n))}{\|u_k + s_ke^+\|^2} \leq \frac{t_k^2}{2}\|e^+\|^2 \to 0$$

which yields $\|v_k^-\| \to 0$ as $k \to \infty$. Hence $1 = \|v_k^- + v_k^0 + t_ke^+\|^2 \to 0$ as $k \to \infty$, which is a contradiction.

Case 2. $t_0 = 0$ and $v^0 \neq 0$. In this case, $v^- = 0$ and $\mathcal{A}v^0 = 0$. By Lemma 2.2, $v^0|_\Lambda \neq 0$. It follows from (f_1) and (3.7) that

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \left[\frac{t_k^2}{2}\|e^+\|^2 - \frac{1}{2}\|v_k^-\|^2 - \sum_{n\in\mathbb{Z}} \frac{F(n, u_k(n) + s_ke^+(n))}{\|u_k + s_ke^+\|^2}\right] \\ &\leq -\liminf_{k\to\infty} \sum_{n\in\Lambda} \frac{F(n, u_k(n) + s_ke^+(n))}{(u_k(n) + s_ke^+(n))^2} \big(v_k^-(n) + v_k^0(n) + t_ke^+(n)\big)^2 \\ &\leq -\sum_{n\in\Lambda} \liminf_{k\to\infty} \frac{F(n, u_k(n) + s_ke^+(n))}{(u_k(n) + s_ke^+(n))^2} (v^0(n))^2 \\ &= -\infty, \end{aligned}$$

which is impossible.

Case 3. $t_0 \neq 0$. In this case, we claim that there exists $\bar{n} \in \Lambda$ such that

$$v^-(\bar{n}) + v^0(\bar{n}) + t_0 e^+(\bar{n}) \neq 0. \tag{3.8}$$

If (3.8) is not true, then one has

$$(v^- + v^0 + t_0 e^+)|_\Lambda = 0. \tag{3.9}$$

It follows from (3.7) that

$$0 \leq 2 \limsup_{k\to\infty} \sum_{n\in\mathbb{Z}} \frac{F(n, u_k(n) + s_k e^+(n))}{\|u_k + s_k e^+\|^2} \leq t_0^2 \|e^+\|^2 - \|v^-\|^2. \tag{3.10}$$

By (2.8) and (3.10), we get

$$\begin{aligned} 0 &\leq t_0^2 \|e^+\|^2 - \|v^-\|^2 \\ &= (\mathcal{A}(v^- + v^0 + t_0 e^+), v^- + v^0 + t_0 e^+)_2 \\ &= \sum_{n\in\mathbb{Z}} \Big[\big|\nabla(v^-(n) + v^0(n) + t_0 e^+(n))\big|^2 + (V(n) - \omega)\big(v^-(n) + v^0(n) + t_0 e^+(n)\big)^2 \Big] \\ &= A_n^1 + A_n^2, \end{aligned} \tag{3.11}$$

where

$$A_n^1 := \sum_{n\in\Lambda} \Big[\big|\nabla(v^-(n) + v^0(n) + t_0 e^+(n))\big|^2 + (V(n) - \omega)\big(v^-(n) + v^0(n) + t_0 e^+(n)\big)^2 \Big]$$

and

$$A_n^2 := \sum_{n\in\mathbb{Z}\setminus\Lambda} \Big[\big|\nabla(v^-(n) + v^0(n) + t_0 e^+(n))\big|^2 + (V(n) - \omega)\big(v^-(n) + v^0(n) + t_0 e^+(n)\big)^2 \Big].$$

By (V_1), (2.8) and (3.4), we obtain $\operatorname{supp} e \subseteq\subseteq \Lambda$ and

$$\begin{aligned} A_n^2 &= \sum_{n\in\mathbb{Z}\setminus\Lambda} \Big|\nabla\Big(v^-(n) + v^0(n) - t_0\big(e^-(n) + e^0(n)\big)\Big)\Big|^2 \\ &\quad + \sum_{n\in\mathbb{Z}\setminus\Lambda} (V(n) - \omega)\big(v^-(n) + v^0(n) - t_0(e^-(n) + e^0(n))\big)^2 \\ &= -\|v^- - t_0 e^-\|^2 - \sum_{n\in\Lambda} \Big|\nabla\Big(v^-(n) + v^0(n) - t_0(e^-(n) + e^0(n))\Big)\Big|^2 \\ &\quad - \sum_{n\in\Lambda} (V(n) - \omega)\Big(v^-(n) + v^0(n) - t_0\big(e^-(n) + e^0(n)\big)\Big)^2. \end{aligned} \tag{3.12}$$

In view of $\operatorname{supp} e \subseteq\subseteq \Lambda$, it follows from (3.4), (3.9), (3.11) and (3.12) that

$$\begin{aligned} 0 &\leq A_n^1 + A_n^2 \\ &= -\|v^- - t_0 e^-\|^2 + t_0 \sum_{n\in\Lambda} \Big[\nabla\big(2(v^-(n) + v^0(n)) + t_0(e^+(n) - e^-(n) - e^0(n))\big)\nabla e(n)\Big] \end{aligned}$$

$$
\begin{aligned}
&+ t_0 \sum_{n\in\Lambda} (V(n)-\omega)\big(2(v^-(n)+v^0(n)) + t_0(e^+(n)-e^-(n)-e^0(n))\big)e(n)\\
&= -\|v^- - t_0 e^-\|^2 - t_0^2 \sum_{n\in\Lambda} \big[\nabla e(n)|^2 + (V(n)-\omega)|e(n)|^2\big]\\
&< 0.
\end{aligned}
$$

This contradiction shows (3.8) holds. By (f_1) and (3.8), we obtain

$$
\begin{aligned}
0 \;&\le\; \limsup_{k\to\infty} \left[\frac{t_k^2}{2}\|e^+\|^2 - \frac{1}{2}\|v_k^-\|^2 - \sum_{n\in\mathbb{Z}} \frac{F(n,u_k(n)+s_k e^+(n))}{\|u_k+s_k e^+\|^2}\right]\\
&\le \frac{t_0^2}{2}\|e^+\|^2 - \liminf_{k\to\infty} \sum_{n\in\Lambda} \frac{F(n,u_k(n)+s_k e^+(n))}{(u_k(n)+s_k e^+(n))^2}\big(v_k^-(n)+v_k^0(n)+t_k e^+(n)\big)^2\\
&\le \frac{t_0^2}{2}\|e^+\|^2 - \sum_{n\in\Lambda} \liminf_{k\to\infty} \frac{F(n,u_k(n)+s_k e^+(n))}{(u_k(n)+s_k e^+(n))^2}\big(v^-(n)+v^0(n)+t_0 e^+(n)\big)^2\\
&= \;-\infty,
\end{aligned}
$$

which is a contradiction. □

When f is periodic in n, the following decomposition property has been proved in [26]. However, without periodic assumption on f, we give a simpler and different proof.

Lemma 3.3. *Assume that (V_0), (f_0), (f_3) and (f_5) are satisfied. If $u_k \rightharpoonup u_0$ in E, then there exists a subsequence denoted again by $\{u_k\}$ such that*

$$
J'(u_k) = J'(u_0) + J'(u_k - u_0) + o(1). \tag{3.13}
$$

Proof. Set $v_k = u_k - u_0$. Then $v_k \rightharpoonup 0$ and $v_k(n) \to 0$ for all $n \in \mathbb{Z}$. To prove (3.13), we only need to show

$$
\sup_{\phi\in E,\,\|\phi\|\le 1} \left|\sum_{n\in\mathbb{Z}} [f(n,u_k(n)) - f(n,u_0(n)) - f(n,v_k(n))]\phi(n)\right| \to 0. \tag{3.14}
$$

Since $u_k \rightharpoonup u_0$ in E and $\|\cdot\|$ is equivalent to $\|\cdot\|_0$, there exists $M_0 > 0$ such that

$$
\max\{\|u_k\|_\infty, \|v_k\|_\infty\} \le M_0.
$$

For any $\varepsilon > 0$, by (f_5), there exists $0 < \delta < 1$ such that

$$
|g(n,s) - g(n,t)| < \varepsilon \tag{3.15}
$$

whenever $|s-t| < \delta$, $s,t \in [-M_0, M_0]$ and $n \in \mathbb{Z}$. Set

$$
\Omega_\delta = \big\{n \in \mathbb{Z} : |u_0(n)| < \delta\big\}, \qquad \Omega_\delta^c = \mathbb{Z} \setminus \Omega_\delta. \tag{3.16}
$$

By (f_5), note that

$$
f(n,u_k) - f(n,v_k) = g(n,u_k)u_0 + [g(n,u_k) - g(n,v_k)]v_k. \tag{3.17}
$$

Denote by $|\Omega_\delta^c|$ the cardinality of Ω_δ^c. By (3.16), it is obvious that

$$
|\Omega_\delta^c| \le \delta^{-2} \sum_{n\in\Omega_\delta^c} |u_0(n)|^2 \le \delta^{-2}\|u_0\|_2^2. \tag{3.18}
$$

For any $\phi \in E$, by (3.15), Hölder inequality and the Sobolev inequality, we obtain

$$\sum_{n\in\Omega_\delta} \big|g(n,u_k(n)) - g(n,v_k(n))\big||v_k(n)||\phi(n)| \le \varepsilon\|v_k\|_2\|\phi\|_2 \le C_2\varepsilon\|\phi\|. \tag{3.19}$$

For any $\phi \in E$, from (f_0), (f_3), Hölder inequality and the Sobolev inequality, one has

$$\begin{aligned}
&\sum_{n\in\Omega_\delta^c} \big|g(n,u_k(n)) - g(n,v_k(n))\big||v_k(n)||\phi(n)| \\
&\le C_3 \sum_{n\in\Omega_\delta^c} \big(1 + |v_k(n)|^{p-2} + |u_0(n)|^{p-2}\big)|v_k(n)||\phi(n)| \\
&\le C_4\Big(\sum_{n\in\Omega_\delta^c} |v_k(n)|^2\Big)^{\frac{1}{2}}\|\phi\| + C_5\Big(\sum_{n\in\Omega_\delta^c} |v_k(n)|^p\Big)^{\frac{p-1}{p}}\|\phi\| \\
&\quad + C_6\Big(\sum_{n\in\Omega_\delta^c} |v_k(n)|^p\Big)^{\frac{1}{p}}\|u_0\|_p^{p-2}\|\phi\|,
\end{aligned}$$

which, together with $v_k \rightharpoonup 0$ and (3.18), implies

$$\sup_{\phi\in E,\,\|\phi\|\le 1} \sum_{n\in\Omega_\delta^c} \big|g(n,u_k(n)) - g(n,v_k(n))\big||v_k(n)||\phi(n)| \to 0, \quad k\to\infty. \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$\sup_{\phi\in E,\,\|\phi\|\le 1} \sum_{n\in\mathbb{Z}} \big|g(n,u_k(n)) - g(n,v_k(n))\big||v_k(n)||\phi(n)| \to 0, \quad k\to\infty. \tag{3.21}$$

By (f_0), (f_3) and standard arguments, it is not difficult to prove that there exists $n_1 \in \mathbb{N}$ such that

$$\sup_{\phi\in E,\,\|\phi\|\le 1} \sum_{|n|>n_1} |g\big(n,u_k(n)\big)||u_0(n)||\phi(n)| < \varepsilon, \quad \forall\, k\in\mathbb{N} \tag{3.22}$$

and

$$\sup_{\phi\in E,\,\|\phi\|\le 1} \Big|\sum_{|n|>n_1} f(n,u_0(n))\phi(n)\Big| < \varepsilon. \tag{3.23}$$

In view of (f_0), we get

$$\sup_{\phi\in E,\,\|\phi\|\le 1} \Big|\sum_{|n|\le n_1} [f(n,u_k(n)) - f(n,u_0(n)) - f(n,v_k(n))]\phi(n)\Big| \to 0, \quad k\to\infty. \tag{3.24}$$

For any $\phi \in E$ with $\|\phi\| \le 1$, it follows from (3.17), (3.21)-(3.24) that

$$\begin{aligned}
&\Big|\sum_{n\in\mathbb{Z}}[f(n,u_k(n)) - f(n,u_0(n)) - f(n,v_k(n))]\phi(n)\Big| \\
&\le \Big|\sum_{|n|\le n_1} [f(n,u_k(n)) - f(n,u_0(n)) - f(n,v_k(n))]\phi(n)\Big| + \Big|\sum_{|n|>n_1} f(n,u_0(n))\phi(n)\Big| \\
&\quad + \Big|\sum_{|n|>n_1} [f(n,u_k(n)) - f(n,v_k(n))]\phi(n)\Big|
\end{aligned}$$

$$\begin{aligned}
&\le \Big|\sum_{|n|\le n_1}[f(n,u_k(n))-f(n,u_0(n))-f(n,v_k(n))]\phi(n)\Big|+\Big|\sum_{|n|>n_1}f(n,u_0(n))\phi(n)\Big|\\
&\quad+\sum_{|n|>n_1}\big|g(n,u_k(n))-g(n,v_k(n))\big||v_k(n)||\phi(n)|+\sum_{|n|>n_1}|g\big(n,u_k(n)\big)||u_0(n)||\phi(n)|\\
&< 2\varepsilon+o(1).
\end{aligned}$$

Since $\varepsilon>0$ is is arbitrary, (3.14) follows from the above inequality. □

Next we shall consider the behaviour of Cerami sequences for J. First we recall that a sequence $\{u_n\}\subset E$ is called a Cerami sequence for J at the level c ($(C)_c$ sequence for short) if $J(u_n)\to c$, $\|J'(u_n)\|(1+\|u_n\|)\to 0$. We say that J satisfies the $(C)_c$ condition if every $(C)_c$ sequence has a convergent subsequence. We have the following result.

Lemma 3.4. *Suppose that (V_0), (VF), (f_0), (f_1), (f_3), (f_4) and (f_5) are satisfied. Then the functional J satisfies the $(C)_c$ condition for any $c>0$.*

Proof. Let $\{u_k\}\subset E$ be a Cerami sequence of J, that is,

$$J(u_k)\to c>0,\qquad \|J'(u_k)\|(1+\|u_k\|)\to 0\qquad k\to\infty. \tag{3.25}$$

Observe that for k large

$$c+1\ge J(u_k)-\frac{1}{2}\langle J'(u_k),u_k\rangle=\sum_{n\in\mathbb{Z}}\mathcal{F}(n,u_k(n)). \tag{3.26}$$

To prove the boundedness of $\{u_k\}$, arguing by contradiction, suppose that $\|u_k\|\to\infty$ as $k\to\infty$. Let $v_k=u_k/\|u_k\|$. Then $\|v_k\|=1$. Passing to a subsequence, we may assume that $v_k\rightharpoonup v$ in E and $v_k(n)\to v(n)$ for all $n\in\mathbb{Z}$. Since the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_0$, there exists $M_1>0$ such that $\max\big\{\|v_k\|_0,\|v_k^+\|_0\big\}\le M_1$. Next we consider two possible cases.

Case 1. $v=0$. By (f_3), there exists $\delta_0>0$ small enough such that

$$\Big|\frac{f(n,s)}{s}\Big|<\frac{1}{4M_1^2} \tag{3.27}$$

for $0<|s|<\delta_0$ and $n\in\mathbb{Z}$. From (V_0) and (1.3), there exists $n_2\in\mathbb{Z}^+$ such that

$$V(n)>\omega,\qquad |n|>n_2 \tag{3.28}$$

and

$$\Big|\frac{f(n,s)}{(V(n)-\omega)s}\Big|<\frac{1}{4M_1^2} \tag{3.29}$$

for $\delta_0\le|s|\le R_0$ and $|n|>n_2$, where R_0 is given in (f_4). For each $k\in\mathbb{N}$, define

$$B_k^1=\{n\in\mathbb{Z}:\,0<|u_k(n)|<\delta_0\},\qquad B_k^2=\{n\in\mathbb{Z}:\,|u_k(n)|>R_0\}$$

and

$$B_k^3=\{|n|\le n_2:\,\delta_0\le|u_k(n)|\le R_0\},\qquad B_k^4=\{|n|>n_2:\,\delta_0\le|u_k(n)|\le R_0\}.$$

For any $k\in\mathbb{N}$, by (2.3), (3.27) and Hölder inequality, we have

$$\frac{1}{\|u_k\|}\sum_{n\in B_k^1}f(n,u_k(n))v_k^+(n)\le\sum_{n\in B_k^1}\Big|\frac{f(n,u_k(n))}{u_k(n)}\Big||v_k(n)v_k^+(n)|\le\frac{1}{4M_1^2}\|v_k\|_2\|v_k^+\|_2\le\frac{1}{4}. \tag{3.30}$$

Since $\nu \in (1, 2/(1-\sigma)]$, we can find $q \geq 2/\sigma$ and $r \geq 2$ such that

$$\frac{1}{\nu} + \frac{1}{q} + \frac{1}{r} = 1. \tag{3.31}$$

It follows from (f_4), (2.1), (3.26), (3.31), Hölder inequality and the Sobolev inequality that

$$\begin{aligned}
\frac{1}{\|u_k\|} \sum_{n\in B_k^2} f(n, u_k(n)) v_k^+(n) &\leq \frac{1}{\|u_k\|^{1-\sigma}} \sum_{n\in B_k^2} \frac{|f(n, u_k(n))|}{|u_k(n)|^\sigma} |v_k(n)|^\sigma |v_k^+(n)| \\
&\leq \frac{1}{\|u_k\|^{1-\sigma}} \left[\sum_{n\in B_k^2} \Big(\frac{|f(n, u_k(n))|}{|u_k(n)|^\sigma} \Big)^\nu \right]^{\frac{1}{\nu}} \|v_k\|_{q\sigma}^\sigma \|v_k^+\|_r \\
&\leq \frac{C_7}{\|u_k\|^{1-\sigma}} \Big[\sum_{n\in B_k^2} \mathcal{F}(n, u_k(n)) \Big]^{\frac{1}{\nu}} \\
&= o(1).
\end{aligned} \tag{3.32}$$

Since $v_k \rightharpoonup 0$ in E, $v_k(n)$ and $v_k^+(n)$ converge to 0 pointwise for all $n \in \mathbb{Z}$. By (f_0), we obtain

$$\begin{aligned}
\frac{1}{\|u_k\|} \sum_{n\in B_k^3} f(n, u_k(n)) v_k^+(n) &\leq \sum_{n\in B_k^3} \Big| \frac{f(n, u_k(n))}{u_k(n)} \Big| |v_k(n) v_k^+(n)| \\
&\leq C_8 \sum_{|n|\leq n_2} |v_k(n) v_k^+(n)| \\
&= o(1).
\end{aligned} \tag{3.33}$$

It follows from (2.3), (3.29) and Hölder inequality that

$$\begin{aligned}
\frac{1}{\|u_k\|} \sum_{n\in B_k^4} f(n, u_k(n)) v_k^+(n) &\leq \sum_{n\in B_k^4} \Big| \frac{f(n, u_k(n))}{(V(n)-\omega) u_k(n)} \Big| \big| (V(n)-\omega) v_k(n) v_k^+(n) \big| \\
&\leq \frac{1}{4M_1^2} \|v_k\|_0 \|v_k^+\|_0 \\
&\leq \frac{1}{4}, \quad \forall\ k \in \mathbb{N}.
\end{aligned} \tag{3.34}$$

By (3.30), (3.32), (3.33) and (3.34), we get

$$\begin{aligned}
1 + o(1) &= \frac{\|u_k\|^2 - \|u_k^-\|^2 - \|u_k^0\|^2 - \langle J'(u_k), u_k^+ \rangle}{\|u_k\|^2} \\
&= \frac{1}{\|u_k\|} \sum_{n\in\mathbb{Z}} f(n, u_k(n)) v_k^+(n) \\
&\leq \sum_{i=1}^4 \sum_{n\in B_k^i} \Big| \frac{f(n, u_k(n))}{u_k(n)} \Big| |v_k(n) v_k^+(n)| \\
&\leq \frac{1}{2} + o(1),
\end{aligned}$$

a contradiction.

Case 2. $v \neq 0$. Let

$$l_0(\mathbb{Z}) = \Big\{u \in l^2(\mathbb{Z}) : \{n \in \mathbb{Z} : |u(n)| > 0\} \text{ is a finite set}\Big\}.$$

It is obvious that $l_0(\mathbb{Z})$ is dense in E. For any $\phi \in l_0(\mathbb{Z})$, it follows from (2.5) and (3.25) that

$$\begin{aligned} o(1) &= \langle J'(u_k), \|u_k\|\phi\rangle \\ &= \|u_k\|\Big[(\mathcal{A}u_k, \phi)_2 - \sum_{n\in\mathbb{Z}} f(n, u_k(n))\phi(n)\Big] \\ &= \|u_k\|^2(\mathcal{A}v_k, \phi)_2 - \|u_k\| \sum_{n\in\mathbb{Z}} f(n, u_k(n))\phi(n), \end{aligned}$$

which implies

$$(\mathcal{A}v_k, \phi)_2 - \frac{1}{\|u_k\|} \sum_{n\in\mathbb{Z}} f(n, u_k(n))\phi(n) = o(1). \tag{3.35}$$

It follows from (f_0), (f_3), (f_4), (3.26) and Hölder inequality that

$$\begin{aligned} &\frac{1}{\|u_k\|} \sum_{n\in\mathbb{Z}} |f(n, u_k(n))\phi(n)| \\ &\leq \frac{1}{\|u_k\|^{1-\sigma}} \sum_{u_k\neq 0} \frac{|f(n, u_k(n))|}{|u_k(n)|^\sigma} |v_k(n)|^\sigma |\phi(n)| \\ &\leq \frac{1}{\|u_k\|^{1-\sigma}} \left[\sum_{0<|u_k|\leq R_0} \frac{|f(n, u_k(n))|}{|u_k(n)|^\sigma} |v_k(n)|^\sigma |\phi(n)| + \sum_{|u_k|>R_0} \frac{|f(n, u_k(n))|}{|u_k(n)|^\sigma} |v_k(n)|^\sigma |\phi(n)|\right] \\ &\leq \frac{1}{\|u_k\|^{1-\sigma}} \left\{C_9 \|v_k\|_2^\sigma \|\phi\|_{2/(2-\sigma)} + \left[\sum_{|u_k|>R_0} \Big(\frac{|f(n, u_k(n))|}{|u_k(n)|^\sigma}\Big)^\nu\right]^{\frac{1}{\nu}} \|v_k\|_{q\sigma}^\sigma \|\phi\|_r\right\} \\ &\leq \frac{C_{10}}{\|u_k\|^{1-\sigma}} \left[\|\phi\|_{2/(2-\sigma)} + \Big(\sum_{|u_k|>R_0} \mathcal{F}(n, u_k(n))\Big)^{\frac{1}{\nu}} \|\phi\|_r\right] \\ &\leq \frac{C_{11}}{\|u_k\|^{1-\sigma}} \big(\|\phi\|_{2/(2-\sigma)} + \|\phi\|_r\big) \\ &= o(1), \end{aligned} \tag{3.36}$$

where ν, q and r are given in (3.31). Combining (3.35) and (3.36), we have

$$(\mathcal{A}v_k, \phi)_2 = o(1), \quad \forall\ \phi \in l_0(\mathbb{Z}). \tag{3.37}$$

Since $v_k \rightharpoonup v$ in E, by (3.37), we get $(\mathcal{A}v, \phi)_2 = 0$, $\forall\, \phi \in l_0(\mathbb{Z})$. This shows that $\mathcal{A}v = 0$. By Lemma 2.2, $v|_\Lambda \neq 0$. It follows from (f_1), (2.9) and Fatou's lemma that

$$0 = \lim_{k\to\infty} \frac{J(u_k)}{\|u_k\|^2}$$

$$\begin{aligned}
&\leq \frac{1}{2} - \liminf_{k\to\infty} \sum_{n\in\mathbb{Z}} \frac{F(n,u_k(n))}{u_k^2(n)} v_k^2(n) \\
&\leq \frac{1}{2} - \sum_{n\in\Lambda} \liminf_{k\to\infty} \frac{F(n,u_k(n))}{u_k^2(n)} v^2(n) \\
&= -\infty,
\end{aligned}$$

which is a contradiction. Thus $\{u_k\}$ is bounded in E.

Next we show that there is a convergent subsequence of $\{u_k\}$. In fact, there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that $\{u_k\}$ weakly converges to some $u_0 \in E$. By standard argument, $J'(u_0) = 0$. Set $\bar{u}_k = u_k - u_0$. Then $\bar{u}_k \rightharpoonup 0$ in E. By equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_0$, there exists $M_2 > 0$ such that

$$\max\big\{\|\bar{u}_k\|_0, \|\bar{u}_k\|_\infty\big\} \leq M_2. \tag{3.38}$$

Since $\dim(E^- \oplus E^0) < \infty$, we have $\|\bar{u}_k^-\| \to 0$ and $\|\bar{u}_k^0\| \to 0$. By Lemma 3.3 and (3.25), we obtain $J'(\bar{u}_k) \to 0$ in E^{-1}. Next we only need to prove $\|\bar{u}_k^+\| \to 0$. For any given $\varepsilon > 0$, by (f_3), there exists $\delta_1 > 0$ such that

$$\Big|\frac{f(n,s)}{s}\Big| < \frac{\varepsilon}{4M_2^2} \tag{3.39}$$

for $0 < |s| < \delta_1$ and all $n \in \mathbb{Z}$. From (V_0) and (1.3), there exists $n_3 \in \mathbb{Z}^+$ such that

$$V(n) > \omega, \quad |n| > n_3 \tag{3.40}$$

and

$$\Big|\frac{f(n,s)}{(V(n)-\omega)s}\Big| < \frac{\varepsilon}{4M_2^2} \tag{3.41}$$

for $\delta_1 \leq |s| \leq M_2$ and $|n| > n_3$. For each $k \in \mathbb{N}$, define

$$\Theta_k^1 = \big\{|n| > n_3 : 0 < |\bar{u}_k(n)| < \delta_1\big\}, \quad \Theta_k^2 = \big\{|n| > n_3 : \delta_1 \leq |\bar{u}_k(n)| \leq M_2\big\}.$$

It follows from (2.3), (3.38) and (3.39) that

$$\Big|\sum_{n\in\Theta_k^1} f(n,\bar{u}_k(n))\bar{u}_k(n)\Big| \leq \sum_{n\in\Theta_k^1} \Big|\frac{f(n,\bar{u}_k(n))}{\bar{u}_k(n)}\Big||\bar{u}_k(n)|^2 < \frac{\varepsilon}{4}, \quad \forall\ k \in \mathbb{N}. \tag{3.42}$$

In view of (2.3), (3.38), (3.40) and (3.41), we obtain

$$\Big|\sum_{n\in\Theta_k^2} f(n,\bar{u}_k(n))\bar{u}_k(n)\Big| \leq \sum_{n\in\Theta_k^2} \Big|\frac{f(n,\bar{u}_k(n))}{(V(n)-\omega)\bar{u}_k(n)}\Big||(V(n)-\omega)\bar{u}_k^2(n)| < \frac{\varepsilon}{4}, \quad \forall\ k \in \mathbb{N}. \tag{3.43}$$

Since $\bar{u}_k \rightharpoonup 0$ in E, $\bar{u}_k(n)$ converges to 0 pointwise for all $n \in \mathbb{Z}$. By (f_0) and (3.38), there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{|n|\leq n_3} \big|f(n,\bar{u}_k(n))\bar{u}_k(n)\big| \leq C_{12} \sum_{|n|\leq n_3} |\bar{u}_k(n)| < \frac{\varepsilon}{2}, \quad k > k_0. \tag{3.44}$$

In combination with (3.42), (3.43) and (3.44), one has

$$\Big|\sum_{n\in\mathbb{Z}} f(n,\bar{u}_k(n))\bar{u}_k(n)\Big| \leq \sum_{|n|\leq n_3} \big|f(n,\bar{u}_k(n))\bar{u}_k(n)\big| + \sum_{i=1}^{2} \Big|\sum_{n\in\Theta_k^i} f(n,\bar{u}_k(n))\bar{u}_k(n)\Big| < \varepsilon, \quad k > k_0,$$

which implies that

$$\lim_{k\to\infty}\Big|\sum_{n\in\mathbb{Z}} f(n,\bar{u}_k(n))\bar{u}_k(n)\Big| = 0. \tag{3.45}$$

From (2.10) and (3.45), we obtain

$$\|\bar{u}_k^+\|^2 \le \big|\langle J'(\bar{u}_k),\bar{u}_k\rangle\big| + \|\bar{u}_k^-\|^2 + \Big|\sum_{n\in\mathbb{Z}} f(n,\bar{u}_k(n))\bar{u}_k(n)\Big| = o(1).$$

Remembering $\|\bar{u}_k^-\|\to 0$ and $\|\bar{u}_k^0\|\to 0$, we get $\|\bar{u}_k\|\to 0$, which means $u_k\to u_0$ in E. Hence J satisfies the $(C)_c$ condition for any $c>0$. □

Proof. [Proof of Theorem 1.1] In order to apply Lemma 2.1 we choose $\varphi = J$, $X = E$, $Y = E^-\oplus E^0$ and $Z = E^+$, where $E^\pm$ and E^0 are given by (2.6). In view of Lemmas 2.1, 3.1, 3.2, 3.4, J has a nonzero critical point which is a nontrivial solution of problem (1.2). □

4. Existence of infinitely many solutions

In this section we obtain infinitely many solutions for problem (1.2) under the symmetry assumption that $f(n,-s) = -f(n,s)$ for all $(n,s)\in\mathbb{Z}\times\mathbb{R}$. First we recall a multiplicity theorem for even functionals which will be used in our proof. Let $X = X^-\oplus X^+$ be a separable Hilbert space with $d = \dim X^- < \infty$. and let $\varphi\in C^1(X,\mathbb{R})$ satisfy

(W1) $\varphi(0) = 0$ and $\exists\, r,\rho > 0$ such that $\varphi(u) > 0$, $\forall\, u\in(B_r\backslash\{0\})\cap X^+$ and $\varphi(u)\ge\rho$, $\forall\, u\in S_r\cap X^+$, where $B_r := \{u\in X : \|u\|\le r\}$ and $S_r := \partial B_r$;

(W2) for any $m\in\mathbb{N}$, there exists a m-dimensional subspace X_m of X such that $X_m\cap X_+$ is bounded, where $X_+ := \{u\in X : \varphi(u)\ge 0\}$.

Let $\mathcal{E}$ denote the class of all closed subsets A of $X\backslash\{0\}$ with respect to the origin and let $\gamma(A)$ denote the usual Krasnoselskii genus of A. Denoting by B_1 the unit ball of X and S_1 its boundary, we set

$$\mathcal{H} = \big\{h\in C(X,X) : h \text{ is an odd homeomorphism, and } h(B_1)\subset X_+\cup\overline{B}_r\big\}$$

and

$$\Gamma_m = \big\{A\subset\mathcal{B} :\ A \text{ is compact and } \gamma(A\cap h(S_1))\ge m,\ \forall\, h\in\mathcal{H}\big\},$$

where $\mathcal{B} := \big\{A\subset X : A \text{ is closed and symmetric}\big\}$. Next we recall some important properties of Γ_m due to [1, 2].

Lemma 4.1. *If φ is even and satisfies (W1) and (W2) then*

(i) $\Gamma_m\neq\emptyset$ for all m;

(ii) $\Gamma_{m+1}\subset\Gamma_m$;

(iii) if $A\in\Gamma_m$ and $U\in\mathcal{E}$ with $\gamma(U)\le q<m$, then $\overline{A\backslash U}\in\Gamma_{m-q}$;

(iv) if ϕ is an odd homeomorphism in X such that $\phi(u) = u$ for all u with $\varphi(u) < 0$ and $\phi^{-1}(X_+)\subset X_+$, then $\phi(A)\in\Gamma_m$ provided $A\in\Gamma_m$.

Lemma 4.2. *Let $\varphi\in C^1(X,\mathbb{R})$ be even and satisfy conditions (W1) and (W2) above. For each $m > d$, let*

$$b_m = \inf_{A\in\Gamma_m}\max_{u\in A}\varphi(u). \tag{4.1}$$

Then

(i) $b_{m+1} \geq b_m \geq \rho > 0$*;*

(ii) each b_m *is a critical level of* φ*, provided that* $(C)_{b_m}$ *holds;*

(iii) if $b = b_{m+1} = \ldots = b_{m+q}$ *for some integer* $q \geq 1$*, then* $\gamma(\mathcal{K}_b) \geq q$*, provided that* $(C)_b$ *holds, where* $\mathcal{K}_b := \{u \in X : \varphi'(u) = 0, \varphi(u) = b\}$.

Remark 4.1. The existence of infinitely many solutions for problem (1.2) or its generalization has been investigated by some classical multiple critical theorems such as the Symmetry Mountain Pass theorem and Fountain theorem, see [15, 18, 37]. These theorems require the following stronger condition

(W2') for any finite dimensional subspace $\tilde{X} \subset X$, $\tilde{X} \cap X_+$ is bounded, which is difficult to verify in our local super-quadratic case. With the help of (f_2), we can introduce a sequence of nested subspaces $(X_m)_{m\in\mathbb{N}}$ on which (W2) holds. When the functional φ satisfies (PS) condition, the proof of Lemma 4.2 can be found in [1, 2]. It is not difficult to see that the same proof still works under (C) condition.

When (f_1) is replaced by (f_2), we can also get

Lemma 4.3. *Suppose that* (V_0)*,* (VF)*,* (f_0)*,* (f_2)*,* (f_3)*,* (f_4) *and* (f_5) *are satisfied. Then the functional* J *satisfies the* $(C)_c$ *condition for any* $c > 0$.

Proof. Since the proof of Lemma 4.3 follows the same line of argument as that of Lemma 3.4, we only highlight the necessary modifications. For the sake of simplicity, we still use the notations introduced in Lemma 3.4. In Case 2, it follows from (f_0), (f_3), (f_4) and (3.25) that (3.37) holds. Then (3.37), together with the fact that $v_k \rightharpoonup v$ in E, implies that $\mathcal{A}v = 0$. By (f_2) and Lemma 2.2, $v|_\Theta \neq 0$. In view of (f_2), (2.9) and Fatou's lemma, one has

$$0 = \lim_{k\to\infty} \frac{J(u_k)}{\|u_k\|^2} \leq \frac{1}{2} - \liminf_{k\to\infty} \sum_{n\in\mathbb{Z}} \frac{F(n, u_k(n))}{u_k^2(n)} v_k^2(n) \leq \frac{1}{2} - \sum_{n\in\Theta} \liminf_{k\to\infty} \frac{F(n, u_k(n))}{u_k^2(n)} v^2(n) = -\infty,$$

which is a contradiction. Thus $\{u_k\}$ is bounded in E. The rest of the proof is identical to that of Lemma 3.4. □

By (f_2), $\Theta \subset \mathbb{Z}$ is an infinite set, there exists a strictly increasing sequence or a strictly decreasing sequence $\{n_k\}_{k=1}^{\infty} \subset \Theta$. Define

$$\phi_i(n) = \begin{cases} 1, & n = n_i, \\ 0, & n \neq n_i. \end{cases}$$

Set $E_m = \operatorname{span}\{\phi_1, \ldots, \phi_m\}$. Then we have the following result.

Lemma 4.4. *Suppose that* (V_0)*,* (f_0)*,* (f_2) *and* (f_3) *are satisfied. Then there exists* $R_m > 0$ *such that*

$$J(u) \leq 0, \quad u \in E_m, \quad \|u\| \geq R_m.$$

Proof. Arguing indirectly, assume that for some sequence $\{u_k\} \subset E_m$ with $\|u_k\| \to \infty$ as $k \to \infty$ such that $J(u_k) > 0$ for all $k \in \mathbb{N}$. Set $v_k = u_k/\|u_k\|$, then $\|v_k\| = 1$. Passing to subsequence, we may assume $v_k \rightharpoonup v$ in E. Since E_m is a finite dimensional space, then $v_k \to v$ and $\|v\| = 1$. Set $\Pi = \{n \in \mathbb{Z} : v(n) \neq 0\}$ and $\Theta_m = \{n_1, n_2, \ldots, n_m\}$. Then $\Pi \neq \emptyset$ and $\Pi \subset \Theta_m$. Moreover,

$$\lim_{k\to\infty} |u_k(n)| = \infty, \quad n \in \Pi. \tag{4.2}$$

By (f_0), (f_2), (f_3), (2.9), (4.2) and Fatou's lemma,

$$\begin{aligned} 0 &\le \limsup_{k\to\infty}\frac{J(u_k)}{\|u_k\|^2} \le \limsup_{k\to\infty}\left[\frac{1}{2}-\sum_{n\in\Pi}\frac{F(n,u_k(n))}{\|u_k\|^2}\right] \\ &\le \frac{1}{2}-\liminf_{k\to\infty}\sum_{n\in\Pi}\frac{F(n,u_k(n))}{|u_k(n)|^2}|v_k(n)|^2 \\ &\le \frac{1}{2}-\sum_{n\in\Pi}\liminf_{k\to\infty}\frac{F(n,u_k(n))}{|u_k(n)|^2}|v(n)|^2 \\ &= -\infty, \end{aligned}$$

a contradiction. □

Next we give the proof of Theorem 1.2. To this end, we need some notations. For $c_2 \ge c_1 > -\infty$, we put

$$\begin{aligned} &J^{c_2} := \{u\in E : J(u)\le c_2\}, \quad J_{c_1} := \{u\in E : J(u)\ge c_1\}, \quad J_{c_1}^{c_2} := J^{c_2}\cap J_{c_1}; \\ &\mathcal{K} := \{u\in E : J'(u)=0\}, \qquad \mathcal{K}_c := \{u\in\mathcal{K} : J(u)=c\}. \end{aligned}$$

Proof. [Proof of Theorem 1.2] In order to apply Lemma 4.2 we choose $\varphi = J$, $X = E$, $X^+ = E^+$, $X^- = E^-\oplus E^0$ and $X_m = E_m$. With the help of Lemmas 3.1 and 4.4, we can verify the functional J satisfies (W1) and (W2). When $m > \dim(E^-\oplus E^0)$, by Lemmas 4.2 and 4.3, we obtain a sequence $\{b_m\}$ of positive critical values of J.

Next we show $\{b_m\}$ is unbounded. We employ the idea of proving Proposition 9.33 in [23]. Suppose the sequence $\{b_m\}$ is bounded. By (i) in Lemma 4.2, $b_{m+1}\ge b_m$. Then $b_m\to b<\infty$ as $m\to\infty$. If $b_m = b$ for all large m, by (iii) in Lemma 4.2, we know $\gamma(\mathcal{K}_b)=\infty$. But in view of Lemma 4.3, $\mathcal{K}_b$ is compact which implies that $\gamma(\mathcal{K}_b)<\infty$ by Proposition 7.5 in [23]. Thus we get $b > b_m$ for all $m > d$. Set

$$K := \{u\in E :\ b_{d+1}\le J(u)\le b \text{ and } J'(u)=0\}.$$

From Proposition 7.5 in [23], there exists $\delta>0$ such that $\gamma(N_\delta(K)) = \gamma(K) = l$, where $N_\delta(K) := \{u\in E : \operatorname{dist}(u,K)\le\delta\}$. Set $s := \max\{l, d+1\}$. By Lemma 1.1.1 in [22], we can choose $\varepsilon_0 < \frac{b-b_s}{2}$, then there exists a continuous map $\eta : [0,1]\times E\to E$ such that

(a1) $\eta(t,u) = u$ if $t=0$ or $u\notin J_{b-2\varepsilon_0}^{b+2\varepsilon_0}$;

(a2) $\eta(t,\cdot)$ is an odd homeomorphism of E for each $t\in[0,1]$;

(a3) $J(\eta(t,u))\le J(u)$ for all $t\in[0,1]$ and $u\in E$;

(a4) $\eta(t,u)$ is odd in u for each $t\in[0,1]$;

(a5) $\eta\big(1, J^{b+\varepsilon_0}\backslash\mathcal{O}\big)\subset J^{b-\varepsilon_0}$, where $\mathcal{O} := N_\delta(K)$.

We choose $j\in\mathbb{N}$ such that $b_j > b-\varepsilon_0$ and $B\in\Gamma_{j+s}$ such that

$$\max_{u\in B} J(u)\le b+\varepsilon_0. \tag{4.3}$$

By (iii) in Lemma 4.1, we have $\overline{B\backslash\mathcal{O}}\in\Gamma_j$. By (a1), (a2), (a3) and (a4), we can choose $\phi=\eta(1,\cdot)$ in (iv) of Lemma 4.1. Consequently $\eta\big(1,\overline{B\backslash\mathcal{O}}\big)\in\Gamma_j$. From (a5), (4.1) and (4.3), we get

$$b_j \le \max_{u\in\eta\left(1,\overline{B\backslash\mathcal{O}}\right)} J(u)\le b-\varepsilon_0 < b_j,$$

a contradiction. □

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