Journal of Applied Analysis and Computation
Volume 17, Number 3, June 2027, 1846–1862
Website:http://www.jaac-online.com
DOI:10.11948/20260159

# A STUDY OF A COUPLED FRACTIONAL $Q$-DIFFERENCE SYSTEMS*

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**Abstract** This manuscript investigates a class of coupled systems driven by Caputo fractional $q$-difference derivatives in Banach spaces. First, we establish a generalized coupled Gronwall inequality in the context of a $q$-fractional integral. Second, using the nonlinear alternative for condensing operators, we prove the existence of a global solution and demonstrate the compactness of the solution set. An illustrative example in a sequence space is provided to validate the main theoretical results.

**Keywords** Caputo fractional $q$-derivative, measure of noncompactness, coupled system, existence results, Gronwall's inequality.

**MSC(2010)** 26A33, 34A08, 47H08.

## 1. Introduction

Fractional calculus, with its long historical development, continues to attract substantial attention across a wide spectrum of scientific fields, owing to its broad applicability. The precision offered by fractional models has further stimulated interest from areas such as economics, fractal theory, biology, plasma physics, and electromagnetic theory [18]. Various facets of the theory have been explored extensively [3].

The inception of $q$-fractional calculus and $q$-fractional difference equations can be traced to Agarwal's paper [4], which laid the groundwork by formally introducing the $q$-fractional Riemann–Liouville operator and establishing its fundamental properties. Investigations into fractional $q$-difference equations emerged in the early 19th century [9] and have since gained renewed and considerable interest in recent decades. Further important insights, particularly regarding initial and boundary value problems for both $q$-difference and fractional $q$-difference equations, are provided in [20]. Parallel to the development of fractional calculus, the theory of quantum $q$-calculus was systematically initiated by Jackson [15, 16] and deals with difference operators and integrals involving a parameter $q \in (0, 1)$. Merging these two fields has given rise to fractional $q$-difference calculus, which enables the study of differentiation and integration of non-integer order within the quantum calculus framework. This hybrid theory originated in the late 1960s with the pioneering works of Al-Salam [25] and Agarwal [4], and has since evolved into a well-established research area, as documented in the comprehensive monograph [9].

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*The authors would like to thank the anonymous referee for his/her careful reading of the manuscript and for the valuable comments and suggestions that helped to improve the quality of the paper.
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Numerous studies have since focused on the properties of fractional $q$-difference operators and the solvability of associated fractional $q$-difference equations (see, e.g., [23,24,26]). Moreover, it is worth emphasizing that certain phenomena are more accurately modeled by coupled systems, as evidenced in [29–31]. Therefore, the study of coupled $q$-fractional systems is of considerable theoretical and applied interest.

The classical Gronwall inequality, initially proved in [13], is often used to provide a priori bounds for solutions, crucial for establishing global existence and various stability results for nonlinear systems. Numerous scholars have since developed variants with more general kernels, as seen in [28,30]. To verify the Leray–Schauder boundary condition within fixed-point principles and obtain global existence, we require a new generalized Gronwall inequality that incorporates fractional $q$-difference integrals and a suitable norm. This inequality differs substantially from previous versions, including those established in [29], and will play a crucial role in analyzing the considered systems in infinite-dimensional spaces.

In recent years, there has been growing interest in boundary value problems involving the Caputo-type fractional $q$-difference derivative, due to their utility in modeling phenomena with non-local and memory-dependent behavior within discrete or quantum frameworks [17]. Substantial progress regarding the existence, uniqueness, and stability of solutions for such problems has been achieved in the literature [1,2,5,6,27].

Motivated by [8], we investigate the existence of global solutions, as well as certain topological properties of the solution set, for the following coupled fractional $q$-difference system subject to nonhomogeneous boundary conditions:

$$\begin{cases} \mathcal{D}_q^{\varrho_1}\mathfrak{z}_1(\zeta) = \mathfrak{S}_1(\zeta, \mathfrak{z}_1(\zeta), \mathfrak{z}_2(\zeta)), & \zeta \in \mathfrak{J} := [0, T], \\ \mathcal{D}_q^{\varrho_2}\mathfrak{z}_2(\zeta) = \mathfrak{S}_2(\zeta, \mathfrak{z}_1(\zeta), \mathfrak{z}_2(\zeta)), & \zeta \in \mathfrak{J} := [0, T], \\ a_1\mathfrak{z}_1(0) + b_1\mathfrak{z}_1(T) = \varpi_1, \\ a_2\mathfrak{z}_2(0) + b_2\mathfrak{z}_2(T) = \varpi_2, \end{cases} \tag{1.1}$$

where $T > 0$, $0 < \varrho_1, \varrho_2 \leq 1$, $\mathcal{D}_q^{\varrho_i}$ denotes the Caputo fractional $q$-difference derivative, $\mathfrak{S}_i : \mathfrak{J} \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ are given nonlinear functions, $\mathbb{Y}$ is a Banach space, $a_i, b_i \in \mathbb{R}$, $\varpi_i \in \mathbb{Y}$ with $a_i + b_i \neq 0$ for $i = 1, 2$.

In contrast to many previous works, we impose only growth conditions and regularity hypotheses expressed via the Kuratowski measure of noncompactness (MNC). This approach enables us to handle nonlinearities that are neither necessarily compact nor Lipschitz continuous, thereby significantly broadening the class of admissible problems. Our main analytical tools are the nonlinear alternative for condensing operators coupled with recent advances in the theory of measures of noncompactness in Banach spaces [7,10]. As distinctive aspects of our work, we highlight the following:

(1) We generalize Gronwall's inequality to incorporate a fractional $q$-difference integral kernel. This extension is central to establishing a priori bounds for solutions, thereby ensuring the global solvability of the system under consideration.

(2) We establish a new existence result, extending recent work in [8] concerning coupled $q$-fractional systems to a general setting, namely, when the nonlinear forcing terms act on an infinite-dimensional Banach space. This extension is achieved by combining a fixed-point principle of Sadovskii type with Leray–Schauder boundary conditions.

(3) We provide an illustrative example in the tempered sequence space $c_0^\theta$ to demonstrate the applicability of our theoretical findings.

The remainder of the paper is organized as follows. In Section 2, we collect the necessary preliminary material. Section 3 presents a new Gronwall-type inequality for coupled systems with fractional $q$-difference integrals. Section 3 is devoted to establishing the existence and compactness of solutions. Finally, to validate our abstract results, an illustrative coupled $q$-fractional system in the tempered sequence space $c_0^\theta$ is provided in Section 4.

## 2. Preliminaries

Let $C(\mathfrak{J},\mathbb{Y})$ denote the space of all continuous functions $g:\mathfrak{J}\to\mathbb{Y}$ equipped with the usual sup-norm. The product Banach space $\mathbb{F}=C(\mathfrak{J},\mathbb{Y})\times C(\mathfrak{J},\mathbb{Y})$ is endowed with the norm

$$\|(g_1,g_2)\|_\infty=\max\left\{\|g_1\|_\infty,\|g_2\|_\infty\right\},\qquad\text{for all }(g_1,g_2)\in\mathbb{F}. \tag{2.1}$$

The space $L^\infty(\mathfrak{J},\mathbb{R})$ consists of all essentially bounded measurable functions on $\mathfrak{J}$, normed by

$$\|g\|_{L^\infty}=\operatorname*{ess\,sup}_{\zeta\in\mathfrak{J}}|g(\zeta)|=\inf\left\{M>0:|g(\zeta)|\le M\text{ for almost every }\zeta\in\mathfrak{J}\right\}.$$

Furthermore, $L^1(\mathfrak{J},\mathbb{R})$ denotes the space of Lebesgue integrable functions $g:\mathfrak{J}\to\mathbb{R}$ with the norm

$$\|g\|_{L^1}=\int_0^T|g(s)|\,ds.$$

Next, we recall some definitions and properties of fractional $q$-calculus.

For $a\in\mathbb{R}$ and $q\in(0,1)$, we define

$$[a]_q=\frac{1-q^a}{1-q},\quad a\in\mathbb{R}.$$

The $q$-analogue of the power $(a-b)^k$ for $k\in\mathbb{N}$ is given by

$$(a-b)^{(0)}=1,\qquad(a-b)^{(k)}=\prod_{i=0}^{k-1}\left(a-bq^i\right),\quad k\in\mathbb{N},\ a,b\in\mathbb{R}.$$

More generally, for $\alpha\in\mathbb{R}$,

$$(a-b)^{(\alpha)}=a^\alpha\prod_{n=0}^{\infty}\frac{a-bq^n}{a-bq^{\alpha+n}}.$$

Note that if $b=0$, then $a^{(\alpha)}=a^\alpha$.

**Remark 2.1.** If $\alpha>0$ and $a\le b\le\zeta$, then $(\zeta-a)^{(\alpha)}\ge(\zeta-b)^{(\alpha)}$.

**Definition 2.1.** [17] The $q$-gamma function is defined by

$$\Gamma_q(\beta)=\frac{(1-q)^{(\beta-1)}}{(1-q)^{\beta-1}},\quad\beta\in\mathbb{R}\setminus\{0,-1,-2,\ldots\},$$

where $(1-q)^{(\beta-1)}$ denotes the $q$-analogue of the power function.

Observe that

$$\Gamma_q(\beta+1)=[\beta]_q\Gamma_q(\beta).$$

**Definition 2.2.** [17] For $q\in(0,1)$, the $q$-derivative of order $n\in\mathbb{N}$ of a function $f$ is defined by

$$\begin{cases}\left(D_qf\right)(y)=\dfrac{f(y)-f(qy)}{(1-q)y}, & y\neq 0,\\ \left(D_qf\right)(0)=\lim\limits_{y\to 0}\left(D_qf\right)(y),\end{cases}$$

and higher-order $q$-derivatives are given recursively by

$$\left(D_q^0f\right)(y)=f(y),\qquad \left(D_q^nf\right)(y)=D_q\left(D_q^{n-1}f\right)(y),\quad n\in\mathbb{N}.$$

**Definition 2.3.** [17] The $q$-integral of a function $f$ defined on $\{\zeta q^n : n\in\mathbb{N}\}\cup\{0\}$ is given by

$$\left(I_qf\right)(y)=\int_0^y f(\zeta)\,d_q\zeta=y(1-q)\sum_{n=0}^{\infty}f\left(yq^n\right)q^n,\quad y\in[0,b],$$

provided the series converges.

If $a\in[0,b]$ and $f$ is defined on $[0,b]$, its $q$-integral from $a$ to $b$ is defined by

$$\int_a^b f(\zeta)\,d_q\zeta=\int_0^b f(\zeta)\,d_q\zeta-\int_0^a f(\zeta)\,d_q\zeta.$$

Analogous to the definition for derivatives, the operator $I_q^n$ can be defined recursively as

$$\left(I_q^0f\right)(y)=f(y)\quad\text{and}\quad\left(I_q^nf\right)(y)=I_q\left(I_q^{n-1}f\right)(y),\quad n\in\mathbb{N}.$$

**Definition 2.4.** [4] The fractional $q$-integral of Riemann–Liouville type of order $\alpha\geq 0$ for a function $f$ is defined by

$$\left(I_q^0f\right)(y)=f(y)$$

and, for $\alpha>0$,

$$\left(I_q^\alpha f\right)(y)=\frac{1}{\Gamma_q(\alpha)}\int_0^y(y-q\zeta)^{(\alpha-1)}f(\zeta)\,d_q\zeta,\quad y\in[0,b],$$

where $(y-q\zeta)^{(\alpha-1)}$ is the $q$-analogue of the power function.

**Lemma 2.1.** [23] *For $\beta\geq 0$ and $\vartheta\in(-1,+\infty)$, we have*

$$\left(I_q^\beta(\zeta-a)^{(\vartheta)}\right)(\zeta)=\frac{\Gamma_q(\vartheta+1)}{\Gamma_q(\beta+\vartheta+1)}(\zeta-a)^{(\beta+\vartheta)},\quad 0<a<\zeta<b.$$

*In particular,*

$$\left(I_q^\beta 1\right)(\zeta)=\frac{\zeta^{(\beta)}}{\Gamma_q(\beta+1)}.$$

**Definition 2.5.** [24] The fractional $q$-derivative of Riemann-Liouville type of order $\alpha\geq 0$ is defined by

$$\left(D_q^0f\right)(y)=f(y),$$

and

$$\left(D_q^\alpha f\right)(y)=\left(D_q^{[\alpha]}I_q^{[\alpha]-\alpha}f\right)(y),$$

where $[\alpha]$ denotes the smallest integer greater or equal to $\alpha$.

**Definition 2.6.** [24] The Caputo fractional $q$-derivative of order $\alpha \geq 0$ of a function $f$ is defined by

$$\left({}^{c}D_{q}^{0}f\right)(y) = f(y),$$

and

$$\left({}^{c}D_{q}^{\alpha}f\right)(y) = \left(I_{q}^{[\alpha]-\alpha}D_{q}^{[\alpha]}f\right)(y).$$

**Lemma 2.2.** [8] *Let $h \in C(\mathfrak{J}, \mathbb{Y})$. Then the boundary value problem*

$$\left({}^{c}D_{q}^{\alpha}y\right)(\zeta) = h(\zeta), \quad \zeta \in \mathfrak{J} := [0,T], \quad 0 < \alpha \leq 1, \tag{2.2}$$

$$ay(0) + by(T) = c, \tag{2.3}$$

*has a unique solution given by*

$$\begin{aligned} y(\zeta) = {} & \frac{c}{a+b} + \frac{1}{\Gamma_q(\alpha)}\int_0^{\zeta}(\zeta - qs)^{(\alpha-1)}h(s)\,d_qs \\ & - \frac{b}{(a+b)\,\Gamma_q(\alpha)}\int_0^{T}(T - qs)^{(\alpha-1)}h(s)\,d_qs, \quad \zeta \in \mathfrak{J}. \end{aligned} \tag{2.4}$$

Next, we denote by $\mathcal{P}(\mathbb{Y})$ the family of all bounded subsets of $\mathbb{Y}$, and by $\overline{\text{conv}}\mathbb{A}$ the closed convex hull of $\mathbb{A}$.

**Definition 2.7.** [7, 10] The Kuratowski measure of noncompactness (MNC) is the map $\Phi : \mathcal{P}(\mathbb{Y}) \to [0,\infty)$ defined by

$$\Phi(\mathbb{G}) = \inf\{\epsilon > 0 : \mathbb{G} \text{ can be covered by finitely many sets of diameter } \leq \epsilon\}.$$

The following properties hold (for details, see [7, 10, 21]):

1. $\Phi(\mathbb{G}_0) \leq \Phi(\mathbb{G}_1)$ if $\mathbb{G}_0 \subset \mathbb{G}_1$, where $\mathbb{G}_0, \mathbb{G}_1 \in \mathcal{P}(\mathbb{Y})$;
2. $\Phi(\{a\} \cup \mathbb{G}) = \Phi(\mathbb{G})$ for every $a \in \mathbb{Y}$ and $\mathbb{G} \in \mathcal{P}(\mathbb{Y})$;
3. $\Phi(U \cup \mathbb{G}) = \Phi(\mathbb{G})$ for every relatively compact set $U \subset \mathbb{Y}$ and $\mathbb{G} \in \mathcal{P}(\mathbb{Y})$;
4. $\Phi(\mathbb{G}) = \Phi(\overline{\mathbb{G}}) = \Phi(\text{conv}\,\mathbb{G})$ for every $\mathbb{G} \in \mathcal{P}(\mathbb{Y})$;
5. $\Phi(\mu\mathbb{G}) = |\mu|\Phi(\mathbb{G})$ for every $\mu \in \mathbb{R}$ and $\mathbb{G} \in \mathcal{P}(\mathbb{Y})$;
6. $\Phi(\mathbb{G}) = 0$ if and only if $\mathbb{G}$ is relatively compact;
7. $\Phi(\mathbb{G}_2 + \mathbb{G}_1) \leq \Phi(\mathbb{G}_2) + \Phi(\mathbb{G}_1)$ for every $\mathbb{G}_2, \mathbb{G}_1 \in \mathcal{P}(\mathbb{Y})$;
8. For every $\mathbb{G} \in \mathcal{P}(\mathbb{Y})$, there exists a countable set $\widetilde{\mathbb{G}} \subset \mathbb{G}$ such that

$$\Phi(\mathbb{G}) \leq 2\Phi(\widetilde{\mathbb{G}}).$$

**Lemma 2.3.** [14] *Let $\widetilde{\mathbb{G}} = \{w_n\}_{n=1}^{+\infty} \subset \mathcal{P}(C(\mathfrak{J}, \mathbb{Y}))$ be a countable set such that $\|w_n(\zeta)\| \leq \varsigma(\zeta)$ for a.e. $\zeta \in \mathfrak{J}$ and all $n \geq 1$, where $\varsigma \in L^1(\mathfrak{J}, \mathbb{R}_+)$. Then, the function $\zeta \mapsto \Phi(\{w_n(\zeta)\}_{n=1}^{+\infty})$ is integrable and*

$$\Phi\left(\left\{\int_0^{\zeta} w_n(s)\,ds\right\}_{n=1}^{+\infty}\right) \leq 2\int_0^{\zeta}\Phi\left(\widetilde{\mathbb{G}}(s)\right)ds.$$

We recall that a continuous map $\mathcal{W}:\mathbb{Y}\to\mathbb{Y}$ is said to be condensing if for every $\Omega\in\mathcal{P}(\mathbb{Y})$ with $\Phi(\Omega)\neq 0$, we have

$$\Phi(\mathcal{W}(\Omega))<\Phi(\Omega).$$

Equivalently, if $\Phi(\mathcal{W}(\Omega))\geq\Phi(\Omega)$, then $\Phi(\Omega)=0$.

The main tool for proving our main result is the following nonlinear alternative for condensing operators.

**Theorem 2.1.** [22] *Let $\mathbb{Y}$ be a Banach space, and let $\mathbb{U}\subset\mathbb{Y}$ be a closed convex set with $0\in\mathbb{U}$. Let $\mathcal{W}:\mathbb{U}\to\mathbb{U}$ be a condensing map. Then either*

1. *$\mathcal{W}$ has a fixed point in $\mathbb{U}$, or*
2. *the set $\mathbb{W}=\{x\in\mathbb{U}:x=\kappa\mathcal{W}x,\ 0<\kappa<1\}$ is unbounded.*

The previous theorem implies the compactness of the solution set; the proof follows similarly to that given in [7, Theorem 1.6.12].

**Lemma 2.4.** [28, Lemma 3.2] *Let $x\in C(\mathfrak{J},\mathbb{Y})$ satisfy the following inequality for $\zeta\in\mathfrak{J}$:*

$$\|x(\zeta)\|\leq a_1+\int_0^{\zeta}\left(a_2\|x(s)\|^{\lambda_1}+a_4\|x(s)\|^{\lambda_3}\right)ds+\int_0^{T}\left(a_3\|x(s)\|^{\lambda_2}+a_5\|x(s)\|^{\lambda_4}\right)ds,$$

*where $\lambda_1,\lambda_3\in[0,1]$, $\lambda_2,\lambda_4\in[0,1)$, and $a_1,a_2,a_3,a_4,a_5\geq 0$ are constants. Then there exists a constant $N^*>0$ such that*

$$\|x(\zeta)\|\leq N^*\quad \text{for all } \zeta\in\mathfrak{J}.$$

Next, we state a new generalized coupled Gronwall inequality.

**Lemma 2.5.** *Let $x_1,x_2\in C(\mathfrak{J},\mathbb{Y})$ satisfy the following system of inequalities:*

$$\begin{aligned}\|x_1(\zeta)\|\leq a_1&+\int_0^{\zeta}(\zeta-qs)^{(\alpha_1-1)}\left[b_1\|x_1(s)\|^{\vartheta_1}+c_1\|x_2(s)\|^{\vartheta_2}\right]d_qs\\&+\int_0^{T}(T-qs)^{(\alpha_1-1)}\left[d_1\|x_1(s)\|^{\vartheta_1}+e_1\|x_2(s)\|^{\vartheta_2}\right]d_qs,\end{aligned}\tag{2.5}$$

*and*

$$\begin{aligned}\|x_2(\zeta)\|\leq a_2&+\int_0^{\zeta}(\zeta-qs)^{(\alpha_2-1)}\left[b_2\|x_1(s)\|^{\sigma_1}+c_2\|x_2(s)\|^{\sigma_2}\right]d_qs\\&+\int_0^{T}(T-qs)^{(\alpha_2-1)}\left[d_2\|x_1(s)\|^{\sigma_1}+e_2\|x_2(s)\|^{\sigma_2}\right]d_qs,\end{aligned}\tag{2.6}$$

*where $\alpha_1,\alpha_2\in(0,1)$, $\vartheta_i,\sigma_i\in\left[0,1-\frac{1}{p}\right)$ for some $1<p<\min\left\{\frac{1}{1-\alpha_1},\frac{1}{1-\alpha_2}\right\}$, and $a_i,b_i,c_i,d_i$, $e_i\geq 0$, $(i=1,2)$ are constants. Then there exists a constant $N^*>0$ such that*

$$\|x_1(\zeta)\|\leq N^*,\quad \|x_2(\zeta)\|\leq N^*,\quad \text{for all } \zeta\in\mathfrak{J}.\tag{2.7}$$

**Proof.** Define

$$y_1(\zeta)=\begin{cases}1, & \|x_1(\zeta)\|\leq 1,\\ x_1(\zeta), & \|x_1(\zeta)\|>1,\end{cases}\qquad y_2(\zeta)=\begin{cases}1, & \|x_2(\zeta)\|\leq 1,\\ x_2(\zeta), & \|x_2(\zeta)\|>1,\end{cases}$$

and set

$$\vartheta = \max\{\vartheta_1, \vartheta_2\}, \quad \sigma = \max\{\sigma_1, \sigma_2\}, \quad \mu = \max\{\vartheta, \sigma\}, \quad \alpha = \min\{\alpha_1, \alpha_2\},$$
$$\mathbf{B}_i = \max\{b_i, c_i\}, \quad \mathbf{C}_i = \max\{d_i, e_i\}, \quad \mathbf{A}_i = a_i + 1, \qquad (i = 1, 2).$$

From (2.5) and (2.6) we obtain

$$\begin{aligned}\|y_1(\zeta)\| \le \mathbf{A}_1 &+ \mathbf{B}_1 \int_0^\zeta (\zeta - qs)^{(\alpha_1 - 1)} \left[\|y_1(s)\|^\vartheta + \|y_2(s)\|^\vartheta\right] d_q s \\ &+ \mathbf{C}_1 \int_0^T (T - qs)^{(\alpha_1 - 1)} \left[\|y_1(s)\|^\vartheta + \|y_2(s)\|^\vartheta\right] d_q s,\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\|y_2(\zeta)\| \le \mathbf{A}_2 &+ \mathbf{B}_2 \int_0^\zeta (\zeta - qs)^{(\alpha_2 - 1)} \left[\|y_1(s)\|^\sigma + \|y_2(s)\|^\sigma\right] d_q s \\ &+ \mathbf{C}_2 \int_0^T (T - qs)^{(\alpha_2 - 1)} \left[\|y_1(s)\|^\sigma + \|y_2(s)\|^\sigma\right] d_q s.\end{aligned} \tag{2.9}$$

Note the elementary inequality

$$\lambda_1^\mu + \lambda_2^\mu \le 2(\lambda_1 + \lambda_2)^\mu, \qquad 0 \le \mu < 1, \quad \lambda_1, \lambda_2 \ge 0. \tag{2.10}$$

Adding (2.8) and (2.9) gives

$$\begin{aligned}&\|y_1(\zeta)\| + \|y_2(\zeta)\| \\ &\le \mathbf{A}_1 + \mathbf{A}_2 + \int_0^\zeta \left[\mathbf{B}_1(\zeta - qs)^{(\alpha_1 - 1)} + \mathbf{B}_2(\zeta - qs)^{(\alpha_2 - 1)}\right] \left[\|y_1(s)\|^\mu + \|y_2(s)\|^\mu\right] d_q s \\ &\quad + \int_0^T \left[\mathbf{C}_1(T - qs)^{(\alpha_1 - 1)} + \mathbf{C}_2(T - qs)^{(\alpha_2 - 1)}\right] \left[\|y_1(s)\|^\mu + \|y_2(s)\|^\mu\right] d_q s.\end{aligned} \tag{2.11}$$

Observe that for $i = 1, 2$,

$$(\zeta - qs)^{(\alpha_i - 1)} \le (\zeta - qs)^{(\alpha - 1)} \quad \text{whenever } \alpha_i \ge \alpha.$$

Hence,

$$\begin{aligned}\|y_1(\zeta)\| + \|y_2(\zeta)\| \le \mathbf{A}_1 + \mathbf{A}_2 &+ (\mathbf{B}_1 + \mathbf{B}_2) \int_0^\zeta (\zeta - qs)^{(\alpha - 1)} \left[\|y_1(s)\|^\mu + \|y_2(s)\|^\mu\right] d_q s \\ &+ (\mathbf{C}_1 + \mathbf{C}_2) \int_0^T (T - qs)^{(\alpha - 1)} \left[\|y_1(s)\|^\mu + \|y_2(s)\|^\mu\right] d_q s.\end{aligned} \tag{2.12}$$

Applying Hölder's inequality (see [19, Lemma 2.1]) together with (2.10), we obtain

$$\begin{aligned}&\|y_1(\zeta)\| + \|y_2(\zeta)\| \\ &\le \mathbf{A}_1 + \mathbf{A}_2 \\ &\quad + (\mathbf{B}_1 + \mathbf{B}_2)\Big(\int_0^\zeta \left((\zeta - qs)^{(\alpha - 1)}\right)^p d_q s\Big)^{\frac{1}{p}} \Big(\int_0^\zeta \left(\|y_1(s)\|^\mu + \|y_2(s)\|^\mu\right)^{\frac{p}{p-1}} d_q s\Big)^{\frac{p-1}{p}} \\ &\quad + (\mathbf{C}_1 + \mathbf{C}_2)\Big(\int_0^T \left((T - qs)^{(\alpha - 1)}\right)^p d_q s\Big)^{\frac{1}{p}} \Big(\int_0^T \left(\|y_1(s)\|^\mu + \|y_2(s)\|^\mu\right)^{\frac{p}{p-1}} d_q s\Big)^{\frac{p-1}{p}}\end{aligned}$$

$$
\begin{aligned}
&\le \mathbf{A}_1 + \mathbf{A}_2 \\
&\quad + 2(\mathbf{B}_1 + \mathbf{B}_2)\Big( \int_0^{\zeta} \big((\zeta - qs)^{(\alpha-1)}\big)^p \, d_q s\Big)^{\frac{1}{p}} \int_0^{\zeta} \big(\|y_1(s)\| + \|y_2(s)\|\big)^{\frac{\mu p}{p-1}} \, d_q s \\
&\quad + 2(\mathbf{C}_1 + \mathbf{C}_2)\Big( \int_0^{T} \big((T - qs)^{(\alpha-1)}\big)^p \, d_q s\Big)^{\frac{1}{p}} \int_0^{T} \big(\|y_1(s)\| + \|y_2(s)\|\big)^{\frac{\mu p}{p-1}} \, d_q s. \qquad (2.13)
\end{aligned}
$$

Now consider $I(\zeta) = \displaystyle\int_0^{\zeta} \big((\zeta - qs)^{(\alpha-1)}\big)^p \, d_q s$. Using the definition of the $q$-integral,

$$
I(\zeta) = \zeta(1-q) \sum_{k=0}^{\infty} q^k \big[(\zeta - q(\zeta q^k))^{(\alpha-1)}\big]^p = \zeta(1-q) \sum_{k=0}^{\infty} q^k \big[(\zeta(1-q^{k+1}))^{(\alpha-1)}\big]^p.
$$

As $k \to \infty$, $q^{k+1} \to 0$, so $1 - q^{k+1} \to 1$, and the general term behaves like $q^k \big(\zeta^{(\alpha-1)}\big)^p$. Since $0 < q < 1$, the geometric series $\displaystyle\sum_{k=0}^{\infty} q^k$ converges; therefore $I(\zeta)$ converges.

Because $0 < \frac{\mu p}{p-1} < 1$, we can apply Lemma 2.4 to conclude that there exists a constant $N^* > 0$ such that

$$
\|y_1(\zeta)\| + \|y_2(\zeta)\| \le N^* \quad \text{for all } \zeta \in \mathfrak{J}.
$$

Consequently,

$$
\|x_1(\zeta)\| \le \|y_1(\zeta)\| \le N^*, \qquad \|x_2(\zeta)\| \le \|y_2(\zeta)\| \le N^* \quad \text{for all } \zeta \in \mathfrak{J}.
$$

□

## 3. Main results

To establish our main result, we introduce the following assumptions:

(H1) The functions $\mathfrak{S}_1, \mathfrak{S}_2 : \mathfrak{J} \times \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ are continuous.

(H2) There exist constants $0 < \vartheta_{1,1}, \vartheta_{1,2}, \vartheta_{2,1}, \vartheta_{2,2} < 1 - \frac{1}{p}$ for some $1 < p < \min\limits_{1 \le i \le 2} \Big\{ \frac{1}{1-\varrho_i} \Big\}$, and positive constants $\mathfrak{R}_1, \mathfrak{R}_2 > 0$ such that

$$
\begin{aligned}
\|\mathfrak{S}_1(\zeta, \mathfrak{z}_1, \mathfrak{z}_2)\| &\le \mathfrak{R}_1\big(\|\mathfrak{z}_1\|^{\vartheta_{1,1}} + \|\mathfrak{z}_2\|^{\vartheta_{1,2}}\big), \\
\|\mathfrak{S}_2(\zeta, \mathfrak{z}_1, \mathfrak{z}_2)\| &\le \mathfrak{R}_2\big(\|\mathfrak{z}_1\|^{\vartheta_{2,1}} + \|\mathfrak{z}_2\|^{\vartheta_{2,2}}\big),
\end{aligned}
$$

for all $\zeta \in \mathfrak{J}$ and $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{Y}$.

(H3) There exist constants $\eta_{i,j} > 0$ $(i, j = 1, 2)$ such that for every $(\mathbb{U}^1, \mathbb{U}^2) \in \mathcal{P}(\mathbb{F})$,

$$
\begin{aligned}
\Phi\big(\mathfrak{S}_1(\zeta, \mathbb{U}^1(\zeta), \mathbb{U}^2(\zeta))\big) &\le \eta_{1,1}\, \Phi(\mathbb{U}^1(\zeta)) + \eta_{1,2}\, \Phi(\mathbb{U}^2(\zeta)), \quad \zeta \in \mathfrak{J}, \\
\Phi\big(\mathfrak{S}_2(\zeta, \mathbb{U}^1(\zeta), \mathbb{U}^2(\zeta))\big) &\le \eta_{2,1}\, \Phi(\mathbb{U}^1(\zeta)) + \eta_{2,2}\, \Phi(\mathbb{U}^2(\zeta)), \quad \zeta \in \mathfrak{J}.
\end{aligned}
$$

**Theorem 3.1.** *Assume that* (H1)–(H3) *hold. Then system (1.1) possesses at least one global solution. Furthermore, the set of all solutions is compact in* $\mathbb{F}$.

**Proof.** According to Lemma 2.2, define the operator $\mathcal{W} : \mathbb{F} \to \mathbb{F}$ by

$$\mathcal{W}(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta) = \big(\mathcal{W}_1(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta),\ \mathcal{W}_2(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta)\big), \tag{3.1}$$

where for $i = 1, 2$,

$$\begin{aligned}\mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta) = \frac{\varpi_i}{a_i + b_i} &+ \frac{1}{\Gamma_q(\varrho_i)} \int_0^\zeta (\zeta - qs)^{(\varrho_i - 1)} \mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big)\, d_q s \\ &- \frac{b_i}{(a_i + b_i)\Gamma_q(\varrho_i)} \int_0^T (T - qs)^{(\varrho_i - 1)} \mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big)\, d_q s.\end{aligned} \tag{3.2}$$

For $R > 0$, consider the closed ball

$$\mathbb{O}_R = \{(\mathfrak{z}_1, \mathfrak{z}_2) \in \mathbb{F} : \|(\mathfrak{z}_1, \mathfrak{z}_2)\|_{\mathbb{F}} \le R\}.$$

To apply Theorem 2.1, we proceed in several steps.

**Step 1.** $\mathcal{W}$ is continuous.

Let $\big(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n}\big)_{n \ge 0}$ be a sequence in $\mathbb{F}$ such that $(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n}) \xrightarrow[n \to \infty]{} (\mathfrak{z}_1, \mathfrak{z}_2)$. Then for each $\zeta \in \mathfrak{J}$ and $i = 1, 2$,

$$\begin{aligned}&\big\|\mathcal{W}_i(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n})(\zeta) - \mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta)\big\| \\ &\le \frac{1}{\Gamma_q(\varrho_i)} \int_0^\zeta (\zeta - qs)^{(\varrho_i - 1)} \big\|\mathfrak{S}_i\big(s, \mathfrak{z}_{1,n}(s), \mathfrak{z}_{2,n}(s)\big) - \mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big)\big\|\, d_q s \\ &\quad + \frac{|b_i|}{|a_i + b_i|\,\Gamma_q(\varrho_i)} \int_0^T (T - qs)^{(\varrho_i - 1)} \big\|\mathfrak{S}_i\big(s, \mathfrak{z}_{1,n}(s), \mathfrak{z}_{2,n}(s)\big) - \mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big)\big\|\, d_q s.\end{aligned}$$

Using Lemma 2.1 we obtain

$$\begin{aligned}&\big\|\mathcal{W}_i(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n})(\cdot) - \mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\cdot)\big\| \\ &\le \left(1 + \frac{|b_i|}{|a_i + b_i|}\right) \frac{T^{(\varrho_i)}}{\Gamma_q(\varrho_i + 1)} \big\|\mathfrak{S}_i(\cdot, \mathfrak{z}_{1,n}(\cdot), \mathfrak{z}_{2,n}(\cdot)) - \mathfrak{S}_i(\cdot, \mathfrak{z}_1(\cdot), \mathfrak{z}_2(\cdot))\big\|.\end{aligned}$$

Since $\mathfrak{S}_i$ $(i = 1, 2)$ are continuous, the right-hand side tends to zero as $n \to \infty$. Consequently,

$$\big\|\mathcal{W}(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n}) - \mathcal{W}(\mathfrak{z}_1, \mathfrak{z}_2)\big\|_{\mathbb{F}} \xrightarrow[n \to \infty]{} 0,$$

which shows that $\mathcal{W}$ is continuous on $\mathbb{F}$.

**Step 2.** $\mathcal{W}$ maps bounded sets of $\mathbb{F}$ into bounded sets.

Let $(\mathfrak{z}_1, \mathfrak{z}_2) \in \mathbb{O}_R$. For each $\zeta \in \mathfrak{J}$ and $i = 1, 2$,

$$\begin{aligned}\big\|\mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta)\big\| \le \frac{\|\varpi_i\|}{|a_i + b_i|} &+ \frac{1}{\Gamma_q(\varrho_i)} \int_0^\zeta (\zeta - qs)^{(\varrho_i - 1)} \big\|\mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big)\big\|\, d_q s \\ &+ \frac{|b_i|}{|a_i + b_i|\,\Gamma_q(\varrho_i)} \int_0^T (T - qs)^{(\varrho_i - 1)} \big\|\mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big)\big\|\, d_q s.\end{aligned}$$

By (H2),

$$\begin{aligned}\big\|\mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta)\big\| \le \frac{\|\varpi_i\|}{|a_i + b_i|} &+ \frac{\mathfrak{R}_i}{\Gamma_q(\varrho_i)} \int_0^\zeta (\zeta - qs)^{(\varrho_i - 1)} \big(\|\mathfrak{z}_1(s)\|^{\vartheta_{i,1}} + \|\mathfrak{z}_2(s)\|^{\vartheta_{i,2}}\big)\, d_q s \\ &+ \frac{|b_i|\,\mathfrak{R}_i}{|a_i + b_i|\,\Gamma_q(\varrho_i)} \int_0^T (T - qs)^{(\varrho_i - 1)} \big(\|\mathfrak{z}_1(s)\|^{\vartheta_{i,1}} + \|\mathfrak{z}_2(s)\|^{\vartheta_{i,2}}\big)\, d_q s.\end{aligned}$$

Since $\|\mathfrak{z}_j(s)\| \le R$ for $j = 1, 2$,

$$\begin{aligned}\big\|\mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta)\big\| &\le \frac{\|\varpi_i\|}{|a_i + b_i|} + \frac{\mathfrak{R}_i\big(R^{\vartheta_{i,1}} + R^{\vartheta_{i,2}}\big)}{\Gamma_q(\varrho_i)} \int_0^{\zeta} (\zeta - qs)^{(\varrho_i - 1)}\, d_q s \\ &\quad + \frac{|b_i|\, \mathfrak{R}_i\big(R^{\vartheta_{i,1}} + R^{\vartheta_{i,2}}\big)}{|a_i + b_i|\, \Gamma_q(\varrho_i)} \int_0^{T} (T - qs)^{(\varrho_i - 1)}\, d_q s.\end{aligned}$$

Applying Lemma 2.1 yields

$$\big\|\mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta)\big\| \le \frac{\|\varpi_i\|}{|a_i + b_i|} + \left(1 + \frac{|b_i|}{|a_i + b_i|}\right) \frac{\mathfrak{R}_i\big(R^{\vartheta_{i,1}} + R^{\vartheta_{i,2}}\big)\, T^{(\varrho_i)}}{\Gamma_q(\varrho_i + 1)} =: \widehat{\mathfrak{R}}_i.$$

Thus

$$\big\|\mathcal{W}(\mathfrak{z}_1, \mathfrak{z}_2)\big\|_{\mathbb{F}} = \max\big\{\|\mathcal{W}_1\|_\infty, \|\mathcal{W}_2\|_\infty\big\} \le \max\big\{\widehat{\mathfrak{R}}_1, \widehat{\mathfrak{R}}_2\big\} =: \widehat{\mathfrak{R}},$$

so $\mathcal{W}(\mathbb{O}_R) \subset \mathbb{O}_{\widehat{\mathfrak{R}}}$.

**Step 3.** $\mathcal{W}$ maps bounded sets into equicontinuous sets.

Let $(\mathfrak{z}_1, \mathfrak{z}_2) \in \mathbb{O}_R$ and $\zeta_1, \zeta_2 \in \mathfrak{J}$ with $\zeta_1 < \zeta_2$. For $i = 1, 2$,

$$\begin{aligned}&\big\|\mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta_2) - \mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta_1)\big\| \\ &= \frac{1}{\Gamma_q(\varrho_i)} \Big\| \int_0^{\zeta_2} (\zeta_2 - qs)^{(\varrho_i - 1)} \mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big)\, d_q s \\ &\quad - \int_0^{\zeta_1} (\zeta_1 - qs)^{(\varrho_i - 1)} \mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big)\, d_q s \Big\| \\ &\le \frac{1}{\Gamma_q(\varrho_i)} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - qs)^{(\varrho_i - 1)} \big\| \mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big) \big\|\, d_q s \\ &\quad + \frac{1}{\Gamma_q(\varrho_i)} \int_0^{\zeta_1} \Big| (\zeta_2 - qs)^{(\varrho_i - 1)} - (\zeta_1 - qs)^{(\varrho_i - 1)} \Big| \big\| \mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big) \big\|\, d_q s.\end{aligned}$$

Using (H2) and the bound $\|\mathfrak{z}_j(s)\| \le R$,

$$\begin{aligned}\big\|\mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta_2) - \mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta_1)\big\| &\le \frac{\mathfrak{R}_i\big(R^{\vartheta_{i,1}} + R^{\vartheta_{i,2}}\big)}{\Gamma_q(\varrho_i)} \bigg[ \int_{\zeta_1}^{\zeta_2} (\zeta_2 - qs)^{(\varrho_i - 1)}\, d_q s \\ &\quad + \int_0^{\zeta_1} \Big| (\zeta_2 - qs)^{(\varrho_i - 1)} - (\zeta_1 - qs)^{(\varrho_i - 1)} \Big|\, d_q s \bigg].\end{aligned}$$

By Lemma 2.1 we obtain

$$\big\|\mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta_2) - \mathcal{W}_i(\mathfrak{z}_1, \mathfrak{z}_2)(\zeta_1)\big\| \le \frac{\mathfrak{R}_i\big(R^{\vartheta_{i,1}} + R^{\vartheta_{i,2}}\big)}{\Gamma_q(\varrho_i + 1)} \big(\zeta_2^{(\varrho_i)} - \zeta_1^{(\varrho_i)}\big).$$

Hence, as $\zeta_2 \to \zeta_1$, the right-hand side tends to zero uniformly with respect to $(\mathfrak{z}_1, \mathfrak{z}_2) \in \mathbb{O}_R$. Therefore $\mathcal{W}(\mathbb{O}_R)$ is equicontinuous.

**Step 4.** $\mathcal{W}$ is condensing.

For $\mathbb{U}^1 \times \mathbb{U}^2 \subset \mathcal{P}(\mathbb{F})$, we define a MNC $\Psi_{\mathbb{F}}$ by

$$\Psi_{\mathbb{F}}(\mathbb{U}^1 \times \mathbb{U}^2) = \Psi(\mathbb{U}^1) + \Psi(\mathbb{U}^2), \tag{3.3}$$

where for $i = 1, 2$,

$$\Psi(\mathbb{U}^i) = \sup_{\zeta\in\mathfrak{J}} e^{-\aleph\zeta}\,\Phi\big(\mathbb{U}^i(\zeta)\big), \qquad \aleph > 0. \tag{3.4}$$

It is easy to verify that $\Psi_{\mathbb{F}}$ inherits all properties of the Kuratowski MNC.

Next, set

$$\Xi_i(\aleph) := \sup_{\zeta\in[0,T]} \int_0^\zeta (\zeta - qs)^{(\varrho_i-1)} e^{-\aleph(\zeta-s)}\, d_q s, \qquad i = 1, 2.$$

We can choose $\aleph$ large enough so that

$$\widehat{\Xi}_{i,j}(\aleph) := \frac{2\eta_{i,j}}{\Gamma_q(\varrho_i)}\,\Xi_i(\aleph) < \frac{|a_i + b_i|}{3|a_i + b_i| + 3|b_i|}, \qquad i, j = 1, 2. \tag{3.5}$$

Now let $\mathbb{U}^1 \times \mathbb{U}^2 \in \mathcal{P}(\mathbb{F})$ be such that

$$\Psi\big(\mathcal{W}_i(\mathbb{U}^1 \times \mathbb{U}^2)\big) \geq \Psi(\mathbb{U}^1 \times \mathbb{U}^2), \qquad i = 1, 2. \tag{3.6}$$

There exists a countable set $\{(v_{1,n}, v_{2,n})\}_{n=1}^\infty$ with

$$\begin{aligned} v_{i,n}(\zeta) = {} & \frac{\varpi_i}{a_i + b_i} + \frac{1}{\Gamma_q(\varrho_i)} \int_0^\zeta (\zeta - qs)^{(\varrho_i-1)} \mathfrak{S}_i\big(s, \mathfrak{z}_{1,n}(s), \mathfrak{z}_{2,n}(s)\big)\, d_q s \\ & - \frac{b_i}{(a_i + b_i)\Gamma_q(\varrho_i)} \int_0^T (T - qs)^{(\varrho_i-1)} \mathfrak{S}_i\big(s, \mathfrak{z}_{1,n}(s), \mathfrak{z}_{2,n}(s)\big)\, d_q s, \end{aligned}$$

where $\{(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n})\}_{n=1}^\infty \subset \mathbb{F}$. Using the properties of the MNC we obtain

$$\begin{aligned} \Psi\big(\{v_{i,n}\}_{n=1}^\infty\big) \leq {} & \Psi\Big(\Big\{\int_0^\zeta \frac{(\zeta - qs)^{(\varrho_i-1)}}{\Gamma_q(\varrho_i)} \mathfrak{S}_i\big(s, \mathfrak{z}_{1,n}(s), \mathfrak{z}_{2,n}(s)\big)\, d_q s\Big\}_{n=1}^\infty\Big) \\ & + \frac{|b_i|}{|a_i + b_i|} \Psi\Big(\Big\{\int_0^T \frac{(T - qs)^{(\varrho_i-1)}}{\Gamma_q(\varrho_i)} \mathfrak{S}_i\big(s, \mathfrak{z}_{1,n}(s), \mathfrak{z}_{2,n}(s)\big)\, d_q s\Big\}_{n=1}^\infty\Big). \end{aligned}$$

Now we estimate the right-hand side. By (H3), for any $s \leq \zeta$,

$$\begin{aligned} & \Phi\Big(\Big\{\frac{(\zeta - qs)^{(\varrho_i-1)}}{\Gamma_q(\varrho_i)} \mathfrak{S}_i\big(s, \mathfrak{z}_{1,n}(s), \mathfrak{z}_{2,n}(s)\big)\Big\}_{n=1}^\infty\Big) \\ & \leq \sum_{j=1}^2 \frac{(\zeta - qs)^{(\varrho_i-1)}\eta_{i,j}}{\Gamma_q(\varrho_i)} \Phi\big(\{\mathfrak{z}_{j,n}(s)\}_{n=1}^\infty\big) \\ & \leq \sum_{j=1}^2 \frac{(\zeta - qs)^{(\varrho_i-1)}\eta_{i,j}}{\Gamma_q(\varrho_i)} e^{\aleph s} \sup_{0\leq\tau\leq T} e^{-\aleph\tau} \Phi\big(\{\mathfrak{z}_{j,n}(\tau)\}_{n=1}^\infty\big) \\ & = \sum_{j=1}^2 \frac{(\zeta - qs)^{(\varrho_i-1)}\eta_{i,j}}{\Gamma_q(\varrho_i)} e^{\aleph s}\, \Psi\big(\{\mathfrak{z}_{j,n}\}_{n=1}^\infty\big). \end{aligned}$$

Applying Lemma 2.3 yields

$$\begin{aligned} & \Phi\Big(\Big\{\int_0^\zeta \frac{(\zeta - qs)^{(\varrho_i-1)}}{\Gamma_q(\varrho_i)} \mathfrak{S}_i\big(s, \mathfrak{z}_{1,n}(s), \mathfrak{z}_{2,n}(s)\big)\, d_q s\Big\}_{n=1}^\infty\Big) \\ & \leq 2\sum_{j=1}^2 \Psi\big(\{\mathfrak{z}_{j,n}\}_{n=1}^\infty\big)\eta_{i,j} \int_0^\zeta \frac{(\zeta - qs)^{(\varrho_i-1)}}{\Gamma_q(\varrho_i)} e^{\aleph s}\, d_q s. \end{aligned}$$

Multiplying by $e^{-\aleph\zeta}$ and taking the supremum over $\zeta \in [0,T]$ gives

$$\begin{aligned}
&\Psi\Big(\Big\{\int_0^{\zeta} \frac{(\zeta - qs)^{(\varrho_i-1)}}{\Gamma_q(\varrho_i)} \mathfrak{S}_i\big(s, \mathfrak{z}_{1,n}(s), \mathfrak{z}_{2,n}(s)\big)\, d_q s\Big\}_{n=1}^{\infty}\Big) \\
&\le \sum_{j=1}^{2} \Psi\big(\{\mathfrak{z}_{j,n}\}_{n=1}^{\infty}\big) \frac{2\eta_{i,j}}{\Gamma_q(\varrho_i)} \sup_{\zeta\in[0,T]} \int_0^{\zeta} (\zeta - qs)^{(\varrho_i-1)} e^{-\aleph(\zeta-s)}\, d_q s \\
&= \sum_{j=1}^{2} \Psi\big(\{\mathfrak{z}_{j,n}\}_{n=1}^{\infty}\big) \widehat{\Xi}_{i,j}(\aleph).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{|b_i|}{|a_i+b_i|} \Psi\Big(\Big\{\int_0^{T} \frac{(T - qs)^{(\varrho_i-1)}}{\Gamma_q(\varrho_i)} \mathfrak{S}_i\big(s, \mathfrak{z}_{1,n}(s), \mathfrak{z}_{2,n}(s)\big)\, d_q s\Big\}_{n=1}^{\infty}\Big) \\
&\le \frac{|b_i|}{|a_i+b_i|} \sum_{j=1}^{2} \Psi\big(\{\mathfrak{z}_{j,n}\}_{n=1}^{\infty}\big) \widehat{\Xi}_{i,j}(\aleph).
\end{aligned}$$

Combining these estimates we obtain

$$\Psi\big(\{v_{i,n}\}_{n=1}^{\infty}\big) \le \frac{|a_i+b_i|+|b_i|}{|a_i+b_i|} \sum_{j=1}^{2} \Psi\big(\{\mathfrak{z}_{j,n}\}_{n=1}^{\infty}\big) \widehat{\Xi}_{i,j}(\aleph).$$

Now condition (3.5) implies

$$\Psi\big(\{v_{i,n}\}_{n=1}^{\infty}\big) \le \frac{1}{3} \sum_{j=1}^{2} \Psi\big(\{\mathfrak{z}_{j,n}\}_{n=1}^{\infty}\big), \qquad i = 1,2.$$

Therefore, using (3.3),

$$\begin{aligned}
\Psi_{\mathbb{F}}\big(\{(v_{1,n}, v_{2,n})\}_{n=1}^{\infty}\big) &= \Psi\big(\{v_{1,n}\}_{n=1}^{\infty}\big) + \Psi\big(\{v_{2,n}\}_{n=1}^{\infty}\big) \\
&\le \frac{2}{3} \sum_{j=1}^{2} \Psi\big(\{\mathfrak{z}_{j,n}\}_{n=1}^{\infty}\big) \\
&= \frac{2}{3} \Psi_{\mathbb{F}}\big(\{(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n})\}_{n=1}^{\infty}\big).
\end{aligned}$$

If (3.6) holds, then

$$\Psi_{\mathbb{F}}\big(\{(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n})\}_{n=1}^{\infty}\big) \le \frac{2}{3} \Psi_{\mathbb{F}}\big(\{(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n})\}_{n=1}^{\infty}\big),$$

which forces $\Psi_{\mathbb{F}}\big(\{(\mathfrak{z}_{1,n}, \mathfrak{z}_{2,n})\}_{n=1}^{\infty}\big) = 0$. Consequently $\Psi_{\mathbb{F}}(\mathbb{U}^1 \times \mathbb{U}^2) = 0$, showing that $\mathbb{U}^1 \times \mathbb{U}^2$ is relatively compact. Hence $\mathcal{W}$ is condensing with respect $\Psi_{\mathbb{F}}$.

**Step 5.** The set $\mathbb{W}$ defined in Theorem 2.1 (2) is bounded.

Let $(\mathfrak{z}_1, \mathfrak{z}_2) \in \mathbb{F}$ satisfy $(\mathfrak{z}_1, \mathfrak{z}_2) = \kappa \mathcal{W}(\mathfrak{z}_1, \mathfrak{z}_2)$ for some $\kappa \in (0,1)$. Then for each $\zeta \in [0,T]$ and $i = 1,2$,

$$\begin{aligned}
\mathfrak{z}_i(\zeta) = \kappa \Bigg[ &\frac{\varpi_i}{a_i+b_i} + \frac{1}{\Gamma_q(\varrho_i)} \int_0^{\zeta} (\zeta - qs)^{(\varrho_i-1)} \mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big)\, d_q s \\
&- \frac{b_i}{(a_i+b_i)\Gamma_q(\varrho_i)} \int_0^{T} (T - qs)^{(\varrho_i-1)} \mathfrak{S}_i\big(s, \mathfrak{z}_1(s), \mathfrak{z}_2(s)\big)\, d_q s \Bigg].
\end{aligned}$$

By (H2) and Lemma 2.1,

$$\begin{aligned}\|\mathfrak{z}_i(\zeta)\| \leq \frac{\|\varpi_i\|}{|a_i+b_i|} + \frac{\mathfrak{R}_i}{\Gamma_q(\varrho_i)}\int_0^{\zeta}(\zeta-qs)^{(\varrho_i-1)}\big(\|\mathfrak{z}_1(s)\|^{\vartheta_{i,1}}+\|\mathfrak{z}_2(s)\|^{\vartheta_{i,2}}\big)\,d_qs \\ + \frac{|b_i|\,\mathfrak{R}_i}{|a_i+b_i|\,\Gamma_q(\varrho_i)}\int_0^{T}(T-qs)^{(\varrho_i-1)}\big(\|\mathfrak{z}_1(s)\|^{\vartheta_{i,1}}+\|\mathfrak{z}_2(s)\|^{\vartheta_{i,2}}\big)\,d_qs.\end{aligned}$$

Applying Lemma 2.5 we conclude that there exists a constant $N^*>0$ such that

$$\|\mathfrak{z}_1(\zeta)\| \leq N^*, \quad \|\mathfrak{z}_2(\zeta)\| \leq N^* \qquad \text{for all } \zeta\in[0,T].$$

Hence

$$\|(\mathfrak{z}_1,\mathfrak{z}_2)\|_{\mathbb{F}} = \max\big\{\|\mathfrak{z}_1\|_\infty, \|\mathfrak{z}_2\|_\infty\big\} \leq N^*,$$

which shows that the set $\mathbb{W}$ is bounded.

All conditions of Theorem 2.1 are satisfied, therefore $\mathcal{W}$ possesses a fixed point. By Step 5, every fixed point of $\mathcal{W}$ is a global solution of system (1.1). □

## 4. An example

In this section, we provide an example to illustrate the applicability of our main result.

Let $c_0$ denote the Banach space of real sequences converging to zero, equipped with the supremum norm:

$$c_0 := \Big\{\mathfrak{z}=(\mathfrak{z}_1,\mathfrak{z}_2,\dots,\mathfrak{z}_n,\dots) : \lim_{n\to\infty}\mathfrak{z}_n=0,\ \|\mathfrak{z}\|_{c_0}=\sup_{n\geq1}|\mathfrak{z}_n|\Big\}.$$

**Definition 4.1.** [11] A sequence $\theta=(\theta_n)$ with positive, nonincreasing terms is called a *tempering sequence*. Denote by $\mathbb{K}$ the linear space of all real sequences $\mathfrak{z}=\{\mathfrak{z}_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}\theta_n\mathfrak{z}_n=0$. This space, denoted by $c_0^\theta$, becomes a Banach space under the norm

$$\|\mathfrak{z}\|_{c_0^\theta} = \sup_{n\geq1}\big\{\theta_n|\mathfrak{z}_n|\big\}.$$

Let $C\big([0,T],c_0^\theta\big)$ be the space of all continuous functions from $[0,T]$ into $c_0^\theta$. It is a Banach space with the norm (see [12])

$$\|\mathfrak{z}\|_{C([0,T],c_0^\theta)} = \max\big\{\|\mathfrak{z}(\zeta)\|_{c_0^\theta} : \zeta\in[0,T]\big\}, \qquad \mathfrak{z}\in C\big([0,T],c_0^\theta\big).$$

Now consider the following coupled system:

$$\begin{cases}\mathcal{D}^{\varrho_1}_{1/3}\mathfrak{z}_1(\zeta) = \mathfrak{S}_1\big(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta)\big), \quad \zeta\in[0,\pi],\\ \mathcal{D}^{\varrho_2}_{1/3}\mathfrak{z}_2(\zeta) = \mathfrak{S}_2\big(\zeta,\mathfrak{z}_1(\zeta),\mathfrak{z}_2(\zeta)\big), \quad \zeta\in[0,\pi],\\ \pi\,\mathfrak{z}_1(0)+e^{\pi}\,\mathfrak{z}_1(\pi) = \big(\frac15,\frac15,\dots,\frac15,\dots\big),\\ \sqrt{\pi}\,\mathfrak{z}_2(0)+\ln(\pi)\,\mathfrak{z}_2(\pi) = \big(13,13,\dots,13,\dots\big),\end{cases} \tag{4.1}$$

which is a particular case of system (1.1) with

$$T=\pi,\quad q=\frac{1}{3},\quad a_1=\pi,\ a_2=\sqrt{\pi},\ b_1=e^{\pi},\ b_2=\ln(\pi),$$
$$\varpi_1=\big(\frac{1}{5},\frac{1}{5},\dots,\frac{1}{5},\dots\big),\qquad \varpi_2=\big(13,13,\dots,13,\dots\big),$$

and $\mathfrak{S}_1,\mathfrak{S}_2:[0,\pi]\times c_0^\theta\times c_0^\theta\to c_0^\theta$ defined by

$$\begin{cases}\mathfrak{S}_1(\zeta,\mathfrak{z}1,\mathfrak{z}2)=\Big\{\mathfrak{S}_{1,n}(\zeta,\mathfrak{z}1,\mathfrak{z}2)\Big\}_{n\geq1}\\ \qquad\qquad=\left\{e^{-\zeta}\left(\displaystyle\sum_{r=n}^{\infty}\frac{\cos\zeta\ |\mathfrak{z}1,n|^{\vartheta_{1,1}}}{4r^2}+\sum_{r=n}^{\infty}\frac{\sin\zeta\ |\mathfrak{z}2,n|^{\vartheta_{1,2}}}{4r^2}\right)\right\}_{n\geq1},\\ \mathfrak{S}_2(\zeta,\mathfrak{z}1,\mathfrak{z}2)=\Big\{\mathfrak{S}_{2,n}(\zeta,\mathfrak{z}1,\mathfrak{z}2)\Big\}_{n\geq1}\\ \qquad\qquad=\left\{(\zeta+1)\left(\displaystyle\sum_{r=n}^{\infty}\frac{|\mathfrak{z}1,n|^{\vartheta_{2,1}}}{4r^2}+\sum_{r=n}^{\infty}\frac{|\mathfrak{z}2,n|^{\vartheta_{2,2}}}{4r^2}\right)\right\}_{n\geq1},\end{cases}$$

for $\zeta\in[0,\pi]$, $\mathfrak{z}1=\{\mathfrak{z}1,n\}_{n\geq1}$, $\mathfrak{z}2=\{\mathfrak{z}2,n\}_{n\geq1}\in c_0^\theta$.

It is clear that (H1) is satisfied. Now choose the tempering sequence $\theta_n=1/n^2$ for all $n\in\mathbb{N}$. For $\mathfrak{z}1,\mathfrak{z}2\in c_0^\theta$ we have

$$\begin{aligned}\big\|\mathfrak{S}_1(\zeta,\mathfrak{z}1,\mathfrak{z}2)\big\|_{c_0^\theta}&=\sup_{n\in\mathbb{N}}\Big\{\theta_n\big|\mathfrak{S}_{1,n}(\zeta,\mathfrak{z}1,\mathfrak{z}2)\big|\Big\}\\&\leq e^{-\zeta}\sup_{n\in\mathbb{N}}\left(\frac{1}{n^2}\sum_{r=n}^{\infty}\frac{1}{4r^2}|\mathfrak{z}1,n|^{\vartheta_{1,1}}+\frac{1}{n^2}\sum_{r=n}^{\infty}\frac{1}{4r^2}|\mathfrak{z}2,n|^{\vartheta_{1,2}}\right)\\&\leq\frac{\pi^2}{24}\Big(\|\mathfrak{z}1\|_{c_0^\theta}^{\vartheta_{1,1}}+\|\mathfrak{z}2\|_{c_0^\theta}^{\vartheta_{1,2}}\Big),\end{aligned}$$

and similarly

$$\big\|\mathfrak{S}_2(\zeta,\mathfrak{z}1,\mathfrak{z}2)\big\|_{c_0^\theta}\leq\frac{\pi^3+\pi}{24}\Big(\|\mathfrak{z}1\|_{c_0^\theta}^{\vartheta_{2,1}}+\|\mathfrak{z}2\|_{c_0^\theta}^{\vartheta_{2,2}}\Big).$$

Thus (H2) holds with

$$\mathfrak{R}_1=\frac{\pi^2}{24},\qquad\mathfrak{R}_2=\frac{\pi^3+\pi}{24}.$$

To verify (H3) we recall that the Kuratowski measure $\Phi$ and the Hausdorff measure $\Lambda$ of noncompactness are related by

$$\Phi(\mathbb{U})\leq2\Lambda(\mathbb{U}),$$

where for a bounded set $\mathbb{U}$ in a Banach space $\mathbb{Y}$,

$$\Lambda(\mathbb{U})=\lim_{m\to\infty}\sup_{\mathfrak{z}\in\mathbb{U}}\big\|(I-P_m)\mathfrak{z}\big\|_{\mathbb{Y}},$$

with $P_m$ being the projection onto the linear span of the first $m$ vectors of the standard basis (see [10]).

Since $c_0$ and $c_0^\theta$ are isometric [11], the Hausdorff MNC on $c_0^\theta$ can be expressed as

$$\Lambda_{c_0^\theta}(\mathbb{U}) = \lim_{n\to\infty} \sup_{\mathfrak{z}\in\mathbb{U}} \left\|(I - P_n)\theta_{\mathfrak{z}}\right\|_\infty. \tag{4.2}$$

For any bounded sets $\mathbb{U}^1, \mathbb{U}^2 \subset c_0^\theta$, using calculations similar to those in [29,31], we obtain

$$\Phi_{c_0^\theta}\big(\mathfrak{S}_i(\zeta, \mathbb{U}^1(\zeta), \mathbb{U}^2(\zeta))\big) \le \eta_{i,1}\Phi_{c_0^\theta}(\mathbb{U}^1(\zeta)) + \eta_{i,2}\Phi_{c_0^\theta}(\mathbb{U}^2(\zeta)), \qquad \zeta \in [0,\pi],\ i = 1,2,$$

with

$$\eta_{1,1} = \eta_{1,2} = \frac{\pi^2}{12}, \qquad \eta_{2,1} = \eta_{2,2} = \frac{\pi^3 + \pi}{12}.$$

Hence (H3) is also satisfied. Consequently, all assumptions of Theorem 3.1 are fulfilled, and therefore system (4.1) possesses at least one global solution; moreover, the set of all solutions is compact in $C([0,\pi], c_0^\theta) \times C([0,\pi], c_0^\theta)$.

## 5. Conclusion

In this manuscript, we have employed the nonlinear alternative for condensing operators to investigate certain topological properties of the set of global solutions for a coupled system driven by Caputo fractional $q$-difference derivatives. The nonlinear terms, which take values in an infinite-dimensional Banach space, are assumed to satisfy only growth conditions and regularity conditions expressed in terms of the Kuratowski measure of noncompactness. These conditions are considerably weaker than the usual Lipschitz or complete continuity assumptions. An illustrative example in the sequence space $c_0^\theta$ has been provided to demonstrate the applicability of the main result.

## Acknowledgements

The authors would like to thank the anonymous referee for his/her careful reading of the manuscript and for the valuable comments and suggestions that helped to improve the quality of the paper.

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Received May 2026; Accepted November 2026; Available online November 2026.