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DISCRETE-TIME SYNCHRONIZATION IN DELAYED QUANTUM-FRACTIONAL OSCILLATOR NETWORKS

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Abstract This study develops a finite-horizon analytical framework for discrete-time nonlinear oscillator networks with delay, based on the (q,τ)-fractional nabla operator. The network dynamics are formulated as Volterra-type summation systems to represent the combined effects of fractional memory, quantum deformation, delayed interactions, graph coupling, and pinning control. Sufficient conditions are analytically derived for existence, uniqueness, continuous dependence on the prescribed initial history, invariance of the synchronization manifold, and bounded transverse disagreement. The resulting conditions explicitly incorporate the fractional order, deformation parameters, delay length, graph structure, coupling strength, and finite simulation horizon. Adaptive pinning is treated through a spectral condition involving the complete controlled network matrix, without assuming unbounded growth of the controller gains. Numerical experiments for three-node delayed networks with fixed and time-varying coupling are supported by convergence criteria, product-truncation error estimates, and recursion-residual checks. A three-regime comparison examines parameters within the analytically supported region, near its sufficient boundary, and outside the guaranteed region. The numerical results indicate smoother finite-horizon convergence behavior and decreasing synchronization disagreement for the tested parameter sets, while also demonstrating the conservative nature of the sufficient bounds. The conclusions are restricted to finite-horizon well-posedness, synchronization invariance, and disagreement control rather than an exact asymptotic synchronization threshold.

Keywords Quantum-deformation fractional calculus, fractional calculus, fractional nabla operator.

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1. Introduction

The study of discrete-time fractional-order systems has attracted significant attention due to their ability to model memory and hereditary effects in complex processes. Unlike classical

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integer-order models, they offer greater flexibility for capturing long-range dependence, making them well suited for applications in control, signal processing, neuroscience, and networked systems. In particular, synchronization in fractional-order networks with time delays has emerged as a key topic, given its relevance in coupled oscillators, multi-agent systems, and brain dynamics. While extensive results exist for continuous-time fractional systems, their discrete-time counterparts remain less explored, especially under generalized operators that incorporate quantum deformation or nonlocal memory effects [20, 22, 25–27].

The study of synchronization in complex dynamical networks has attracted considerable attention due to its wide applications in neuroscience, electrical engineering, secure communications, and coupled physical systems [5, 15, 24]. Classical synchronization approaches, typically based on integer-order differential or difference models, often fail to capture the long-range interactions and memory effects observed in real-world systems [18, 19, 28].

To address these limitations, recent advances in fractional and quantum calculus have introduced powerful tools for modeling such phenomena [4, 13, 14]. In particular, the (q,τ)-calculus extends classical q-calculus by incorporating a time-scaling parameter τ [2, 8], providing a flexible framework for describing systems with nonlocal memory and nonuniform temporal structures. A notable development is the (q,τ)-fractional nabla operator [1, 3, 6, 7, 9, 23], which enables the formulation of discrete-time fractional dynamical systems that effectively capture hereditary and quantum-scale effects [10–12].

The stability of delayed dynamical systems with memory requires specialized analytical tools beyond classical methods [16, 17, 21]. In this work, we develop an extended Lyapunov-Krasovskii functional tailored to the (q,τ)-fractional nabla framework to study synchronization in discrete-time nonlinear networks with delay. The main objective is to derive sufficient conditions for global asymptotic synchronization by explicitly linking the fractional order α, delay length d, coupling structure (via Laplacian eigenvalues), and the deformation parameters (q,τ). We establish a threshold condition on the coupling strength that guarantees convergence of the synchronization error to zero. Theoretical results are validated through numerical simulations of a delayed three-node nonlinear oscillator network, demonstrating that (q,τ)-fractional memory enhances stability and robustness against delays. The findings verify that the network states synchronize in spite of delay-induced disruptions when the parameters are adjusted correctly (Figure 1).

2. Preliminaries

Definition 2.1. Let the parameters $0 < q < 1$, $\tau > 0$. The (q,τ)-Gamma function is defined by

$$\Gamma_{q,\tau}(x) := (1-q)^{1-x} \prod_{k=0}^{\infty} \frac{1-q^{\tau(k+1)}}{1-q^{\tau x+\tau k}}.$$

This simplifies to $q \to 1$ and $\tau \to 1$ and generalizes the standard Gamma function (see [4, 13, 14]). Additionally, the definition of the (q,τ)-fractional binomial coefficient is

$$\binom{\alpha}{k}_{(q,\tau)} := \frac{\Gamma_{q,\tau}(\alpha+1)}{\Gamma_{q,\tau}(k+1)\Gamma_{q,\tau}(\alpha-k+1)}.$$

Definition 2.2. Define the (q,τ)-nabla operator for discrete functions $f(n)$ as:

$$\nabla_{q,\tau} f(n) := \frac{f(n) - f(n+1)}{(n)_{(q,\tau)}},$$

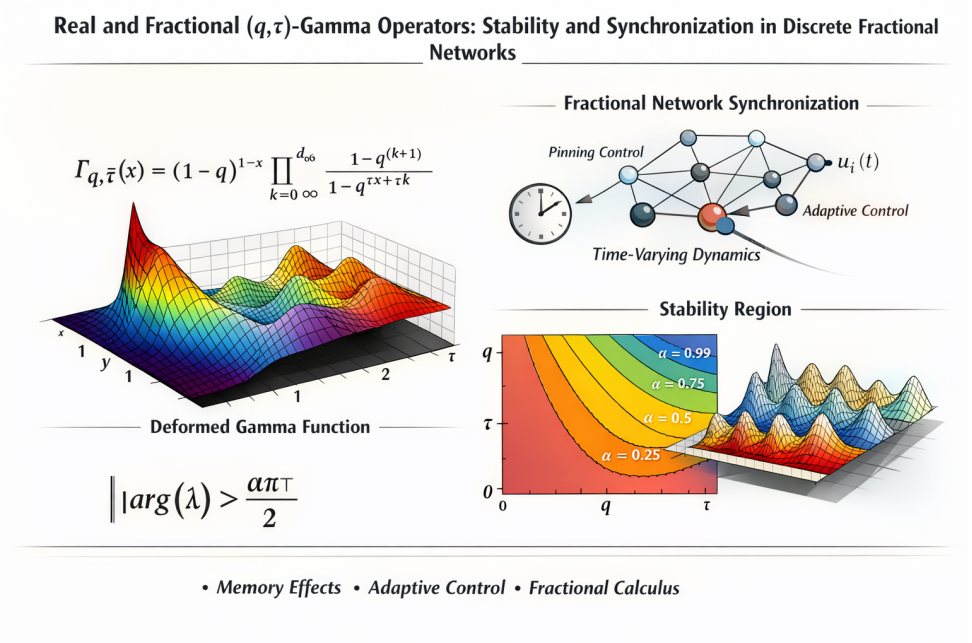

Figure 1. The left panel shows the deformation of the Gamma function $\Gamma_{q,\tau}(x)$, which forms the analytical backbone of the fractional dynamics. The right illustrates stability region boundaries in (q,τ)-space for multiple fractional orders α, and a schematic diagram of network synchronization under adaptive and pinning control. Together, these elements capture the interplay between memory effects, fractional-order deformation, and synchronization behavior in discrete-time dynamical systems.

where $(n)_{q,\tau}$ is a suitable normalization depending on the (q,τ)-difference scale. This allows us to define discrete analogs of integrals and summation using:

$$\sum_{k=0}^{n} f(k) d_{q,\tau} k \approx \sum_{k=0}^{n} f(k) w_k^{(q,\tau)},$$

where $w_k^{(q,\tau)}$ is a weight determined by the (q,τ)-lattice structure. For a fixed deformation pair $0 < q < 1$ and $\tau > 0$, the (q,τ)-integer is defined as

$$(n)_{(q,\tau)} := \frac{1-q^{\tau n}}{1-q^{\tau}} \approx \sum_{k=0}^{n-1} q^{\tau k}, \quad n \in \mathbb{N}_0.$$

Proposition 2.1. *Let $f, g : \mathbb{N}_0 \to \mathbb{R}$, and define the operator*

$$\nabla_{q,\tau} f(n) = \frac{f(n) - f(n+1)}{(n)_{(q,\tau)}}, \quad (n)_{(q,\tau)} = \frac{1-(q\tau)^n}{1-q}, \quad 0 < q < 1,\ \tau > 0.$$

Then the following properties hold:

1. ***Linearity:*** *For any scalars $a, b \in \mathbb{R}$,*

$$\nabla_{q,\tau}(af(n) + bg(n)) = a\nabla_{q,\tau} f(n) + b\nabla^*_{q,\tau} g(n).$$

2. ***Deformed scaling law:*** *If $f(n) = n^{\alpha}$, for $\alpha \in \mathbb{R}$, then*

$$\nabla_{q,\tau} f(n) = \frac{n^{\alpha} - (n+1)^{\alpha}}{(n)_{(q,\tau)}}, \quad \text{which behaves as } -\alpha n^{\alpha-1}(1+o(1)) \text{ as } n \to \infty.$$

3. ***Limit consistency:*** *As $q \to 1^-$ and $\tau \to 1$, we recover the classical backward difference:*

$$\lim_{q\to 1^-,\, \tau\to 1} \nabla_{q,\tau} f(n) = f(n) - f(n+1).$$

4. ***Non-negativity on decreasing functions:*** *If $f(n)$ is decreasing (i.e., $f(n) \geq f(n+1)$), then $\nabla_{q,\tau} f(n) \geq 0$.*

Proof.

(1) Linearity follows directly from the linearity of subtraction and scalar multiplication in the numerator:

$$\begin{aligned}\nabla_{q,\tau}(af(n) + bg(n)) &= \frac{af(n) + bg(n) - af(n+1) - bg(n+1)}{(n)_{(q,\tau)}} \\ &= a\nabla_{q,\tau} f(n) + b\nabla_{q,\tau} g(n).\end{aligned}$$

(2) Substituting $f(n) = n^\alpha$, we get

$$\nabla_{q,\tau} n^\alpha = \frac{n^\alpha - (n+1)^\alpha}{(n)_{(q,\tau)}},$$

which asymptotically behaves like

$$-\alpha n^{\alpha-1}/(n)_{(q,\tau)}.$$

(3) Observe that

$$\lim_{q\to 1^-} (n)_{(q,\tau)} = \lim_{q\to 1^-} \frac{1-(q\tau)^n}{1-q} = n\tau^n,$$

so in the limit $q \to 1, \tau \to 1$, we get

$$\nabla_{q,\tau} f(n) \to f(n) - f(n+1),$$

the classical backward difference (up to unit weight).

(4) If $f(n) \geq f(n+1)$, then $f(n) - f(n+1) \geq 0$, and since $(n)_{(q,\tau)} > 0$, the result follows.

□

We proceed to define the fractional case of the (q,τ)-nabla operator.

Definition 2.3. Let $f : \mathbb{N}_0 \to \mathbb{R}$, and let $0 < q < 1$, $\tau > 0$, $0 < \alpha < 1$. The fractional (q,τ)-nabla operator of order α using the (q,τ)-Gamma function is defined by

$$\nabla^\alpha_{q,\tau} f(n) := \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} \frac{f(k) - f(k+1)}{[(n-k)_{(q,\tau)}]^\alpha}, \tag{2.1}$$

where

$$(n-k)_{(q,\tau)} := \frac{1-(q\tau)^{n-k}}{1-q}, \tag{2.2}$$

and $\Gamma_{q,\tau}(x)$ is the (q,τ)-Gamma function satisfying the recurrence

$$\Gamma_{q,\tau}(x+1) = (x)_{(q,\tau)} \Gamma_{q,\tau}(x), \quad \Gamma_{q,\tau}(1) = 1.$$

3. Stability analysis of discrete fractional systems

The operator $\nabla_{q,\tau}^{\alpha}$ effectively captures memory effects in discrete-time nonlinear systems.

Definition 3.1 ((q,τ)-fractional mild solution)**.** Let $0 < \alpha < 1$, $0 < q < 1$, $\tau > 0$, and $0 < q\tau < 1$. A function $u : \mathbb{N}_N \longrightarrow \mathbb{R}$ is called a finite-horizon mild solution of the (q,τ)-fractional initial-value problem if $u(0) = u_0$ and

$$u(n) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{\alpha-1} f(k, u(k)), \qquad 1 \le n \le N, \tag{3.1}$$

where $(n-k)_{q,\tau}$ is given in Eq. (2.2).

Theorem 3.1 (Finite-horizon well-posedness for the (q,τ)-fractional Volterra equation with kernel of order $\alpha - 1$)**.** *Let $0 < \alpha < 1$, $0 < q < 1$, $\tau > 0$, and assume that $0 < q\tau < 1$. Let $\mathbb{N}_N := \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$. Assume that $\Gamma_{q,\tau}(\alpha) > 0$ and that $f : \mathbb{N}_{N-1} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies*

$$|f(n,x) - f(n,y)| \le \aleph |x - y|, \qquad n \in \mathbb{N}_{N-1}, \quad x, y \in \mathbb{R},$$

for some $\aleph > 0$. Consider the finite-horizon fractional Volterra equation

$$u(n) = u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{\alpha-1} f(k, u(k)), \qquad 1 \le n \le N, \tag{3.2}$$

with $u(0) = u_0$. If

$$K_N := \frac{1}{\Gamma_{q,\tau}(\alpha)} \max_{1 \le n \le N} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{\alpha-1}, \ \aleph K_N < 1, \tag{3.3}$$

then equation (3.2) has a unique solution $u : \mathbb{N}_N \longrightarrow \mathbb{R}$. Moreover, if u and v are the solutions corresponding to the initial values u_0 and v_0, respectively, then

$$\|u - v\|_{\infty,N} \le \frac{|u_0 - v_0|}{1 - \aleph K_N}, \tag{3.4}$$

where $\|u\|_{\infty,N} := \max_{0 \le n \le N} |u(n)|$. If, in addition, $f(n,0) = 0$, $n \in \mathbb{N}_{N-1}$, then $u(n) \equiv 0$ is a solution corresponding to $u_0 = 0$.

Proof. Let $X_N := \{u : \mathbb{N}_N \to \mathbb{R}\}$ be equipped with the norm $\|u\|_{\infty,N} := \max_{0 \le n \le N} |u(n)|$. Since X_N is finite-dimensional, it is a Banach space. Define $T : X_N \to X_N$ by $(Tu)(0) := u_0$ and

$$(Tu)(n) := u_0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{\alpha-1} f(k, u(k)), \qquad 1 \le n \le N.$$

Because $0 < q\tau < 1$, we have $(n-k)_{q,\tau} = \frac{1-(q\tau)^r}{1-q} > 0$. Thus, every kernel term in the definition of T is positive and well-defined. For $u, v \in X_N$ and $1 \le n \le N$, we have

$$|(Tu)(n) - (Tv)(n)| \le \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{\alpha-1} |f(k, u(k)) - f(k, v(k))|$$

$$\leq \frac{\aleph}{\Gamma_{q,\tau}(\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{\alpha-1} |u(k) - v(k)|$$
$$\leq \frac{\aleph}{\Gamma_{q,\tau}(\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{\alpha-1} \|u - v\|_{\infty,N}$$
$$\leq \aleph K_N \|u - v\|_{\infty,N}.$$

At $n = 0$, $|(Tu)(0) - (Tv)(0)| = 0$. Taking the maximum over $0 \leq n \leq N$, we obtain $\|Tu - Tv\|_{\infty,N} \leq \aleph K_N \|u - v\|_{\infty,N}$. Since $\aleph K_N < 1$, the operator T is a contraction. The Banach fixed-point theorem therefore guarantees a unique fixed point $u \in X_N$. The identity $Tu = u$ is precisely (3.2); hence, the problem has a unique solution. Now let u and v correspond to the initial values u_0 and v_0, respectively. For $1 \leq n \leq N$,

$$|u(n) - v(n)| \leq |u_0 - v_0| + \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{\alpha-1} |f(k, u(k)) - f(k, v(k))|$$
$$\leq |u_0 - v_0| + \aleph K_N \|u - v\|_{\infty,N}.$$

The same inequality is valid at $n = 0$. Therefore, $\|u-v\|_{\infty,N} \leq |u_0-v_0|+\aleph K_N \|u-v\|_{\infty,N}$. Since $\aleph K_N < 1$, it follows that $\|u-v\|_{\infty,N} \leq \frac{|u_0-v_0|}{1-\aleph K_N}$. Finally, suppose that $u_0 = 0$ and $f(n, 0) = 0$ for every $n \in \mathbb{N}_{N-1}$. Substituting $u(n) = 0$ into (3.2) gives

$$0 + \frac{1}{\Gamma_{q,\tau}(\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{\alpha-1} f(k, 0) = 0.$$

Hence, $u(n) \equiv 0$ is a solution. Its uniqueness follows from the first part of the theorem. □

We formulate the following result directly for the finite-horizon Volterra equation.

Theorem 3.2 (Finite-horizon well-posedness for the (q, τ)-fractional Volterra equation with kernel of order $-\alpha$)**.** *Let $0 < \alpha < 1$, $0 < q < 1$, $\tau > 0$, $0 < q\tau < 1$, $\mathbb{N}_N := \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$ and $\Gamma_{q,\tau}(1-\alpha) > 0$ Let $f : \mathbb{N}_{N-1} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the uniform Lipschitz condition*

$$|f(n, x) - f(n, y)| \leq \wp |x - y|, \qquad n \in \mathbb{N}_{N-1}, \quad x, y \in \mathbb{R}, \tag{3.5}$$

for some $\wp > 0$. Consider the finite-horizon Volterra equation

$$u(n) = u_0 + \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{-\alpha} f(k, u(k)), \qquad 1 \leq n \leq N, \tag{3.6}$$

with $u(0) = u_0$. Define

$$C_N := \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \max_{1 \leq n \leq N} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{-\alpha}, \ \wp C_N < 1, \tag{3.7}$$

then equation (3.6) possesses a unique mild solution $u : \mathbb{N}_N \longrightarrow \mathbb{R}$. Moreover, if u and v are the solutions corresponding to the initial values u_0 and v_0, respectively, then

$$\|u - v\|_{\infty,N} \leq \frac{|u_0 - v_0|}{1 - \wp C_N}, \tag{3.8}$$

where $\|u\|_{\infty,N} := \max_{0 \leq n \leq N} |u(n)|$. Thus, the mild solution depends continuously on the initial value.

Proof. Define the function space $\mathcal{X} := \{u : \mathbb{N}_0 \to \mathbb{R} \mid \|u\|_\infty < \infty\}$, with the norm $\|u\| := \sup_n |u(n)|$, which is a Banach space. We reformulate the difference equation via an integral-like representation:

$$u(n) = u_0 + \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} \left[(n-k)_{(q,\tau)}\right]^{-\alpha} [f(k, u(k))].$$

Define the operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by:

$$(\mathcal{T}u)(n) := u_0 + \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} \left[(n-k)_{(q,\tau)}\right]^{-\alpha} f(k, u(k)).$$

Let $u, v \in \mathcal{X}$. By the Lipschitz property:

$$|(\mathcal{T}u)(n) - (\mathcal{T}v)(n)| \leq \frac{\wp}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} \left[(n-k)_{(q,\tau)}\right]^{-\alpha} |u(k) - v(k)|.$$

Taking supremum over n, we get:

$$\|\mathcal{T}u - \mathcal{T}v\| \leq \left(\frac{\wp}{\Gamma_{q,\tau}(1-\alpha)} \cdot \sup_n \sum_{k=0}^{n-1} \left[(n-k)_{(q,\tau)}\right]^{-\alpha} \right) \|u - v\|.$$

If the Lipschitz constant $\wp$ satisfies:

$$\wp < \left(\frac{1}{\Gamma_{q,\tau}(1-\alpha)} \cdot \sup_n \sum_{k=0}^{n-1} \left[(n-k)_{(q,\tau)}\right]^{-\alpha} \right)^{-1},$$

then $\mathcal{T}$ is a contraction mapping on $\mathcal{X}$, and by Banach's fixed point theorem, $\mathcal{T}$ has a unique fixed point u, which solves the original equation. □

Remark 3.1 (Distinct roles of Theorems 3.1 and 3.2)**.** Although Theorems 3.1 and 3.2 have similar fixed-point structures, they address two different (q, τ)-fractional Volterra kernels. Theorem 3.1 concerns the kernel with exponent $\alpha - 1$, which is the natural kernel associated with a fractional summation operator of order α. Theorem 3.2 concerns the kernel with exponent $-\alpha$, which corresponds formally to a fractional summation operator of complementary order $1 - \alpha$. Studying both cases is useful because the two kernels assign different weights to the previous states. The $(\alpha - 1)$-order kernel describes memory associated with the fractional order α, whereas the $(-\alpha)$-order kernel describes the complementary memory regime associated with $1 - \alpha$. This distinction is particularly relevant when comparing different formulations of fractional discrete dynamics and when determining which kernel is appropriate for a particular model. Nevertheless, the two equations are treated as distinct finite-horizon Volterra problems. No equivalence between them, or between either Volterra equation and the original fractional difference equation, is assumed unless the corresponding fractional integral and inversion identities are established separately.

Corollary 3.1 (Vector-valued extension)**.** *Under the hypotheses and finite-horizon contraction condition of Theorem 3.2, the corresponding vector-valued Volterra equation has a unique mild solution on $\mathbb{N}_N$, provided that its nonlinear term is uniformly Lipschitz with respect to the chosen norm on $\mathbb{R}^m$. Moreover, the solution depends continuously on its initial vector.*

Proof. Apply the proof of Theorem 3.2 in the finite-dimensional Banach space of $\mathbb{R}^m$-valued sequences, replacing the absolute value by the chosen norm on $\mathbb{R}^m$. The contraction constant remains unchanged. □

4. Finite-horizon synchronization of (q,τ)-Volterra networks

Consider a network consisting of M identical nodes, where the state of node i at time n is $u_i(n) \in \mathbb{R}^d$, $i = 1,\dots,M$. Let $L_G = (\ell_{ij}) \in \mathbb{R}^{M\times M}$ be the symmetric positive-semidefinite Laplacian matrix of an undirected connected graph. We adopt the convention $\ell_{ij} \le 0 \quad (i \ne j)$, $\ell_{ii} = -\sum_{j\ne i}\ell_{ij}$, $L_G\mathbf{1}_M = 0$. Its eigenvalues are ordered as $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_M$. Let $H \in \mathbb{R}^{d\times d}$ be the inner-coupling matrix and let $\kappa \ge 0$ be the coupling strength. We consider the finite-horizon network Volterra system

$$u_i(n) = u_i(0) + \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha}\left[f(u_i(k)) - \kappa\sum_{j=1}^{M}\ell_{ij}Hu_j(k)\right], \tag{4.1}$$

for $i = 1,\dots,M$, $n = 1,\dots,N$. Here, $0 < \alpha < 1$, $0 < q < 1$, $\tau > 0$, $0 < q\tau < 1$, $(r)_{q,\tau} = \frac{1-(q\tau)^r}{1-q}$. The sign in (4.1) is consistent with the positive-semidefinite Laplacian convention. Indeed, the coupling term is $-\kappa\sum_{j=1}^{M}\ell_{ij}Hu_j$, or, equivalently, $\kappa\sum_{j\ne i}a_{ij}H(u_j-u_i)$, $a_{ij} := -\ell_{ij} \ge 0$. Thus, both representations describe diffusive coupling. Define $U(n) := \operatorname{col}\big(u_1(n),\dots,u_M(n)\big) \in \mathbb{R}^{Md}$ and $\mathcal{F}(U) := \operatorname{col}\big(f(u_1),\dots,f(u_M)\big)$. Then (4.1) can be written as

$$U(n) = U(0) + \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha}\left[\mathcal{F}(U(k)) - \kappa(L_G\otimes H)U(k)\right]. \tag{4.2}$$

This representation fixes the Laplacian sign convention and will be used throughout the subsequent analysis and numerical examples.

Theorem 4.1 (Finite-horizon well-posedness, synchronization invariance, and disagreement estimate)**.** *Let $M \ge 2$ be the number of nodes and let $d \ge 1$ be the dimension of each node state. Suppose that $0 < \alpha < 1$, $0 < q < 1$, $\tau > 0$, $0 < q\tau < 1$, and let $\mathbb{N}_N := \{0,1,\dots,N\}$, $N \in \mathbb{N}, \Gamma_{q,\tau}(1-\alpha) > 0$. Let $L_G = (\ell_{ij}) \in \mathbb{R}^{M\times M}$ be the symmetric positive-semidefinite Laplacian matrix of an undirected connected graph. Thus, $L_G\mathbf{1}_M = 0$ and its eigenvalues satisfy $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_M$. Let $H \in \mathbb{R}^{d\times d}$ be the inner-coupling matrix and let $\kappa \ge 0$ be the coupling strength. Suppose that $f : \mathbb{R}^d \to \mathbb{R}^d$ is globally Lipschitz:*

$$\|f(x) - f(y)\|_2 \le \mathcal{L}_f\|x-y\|_2, \qquad x,y \in \mathbb{R}^d, \tag{4.3}$$

for some $\mathcal{L}_f \ge 0$. Consider the finite-horizon coupled Volterra system

$$\begin{aligned} u_i(n) = u_i(0) &+ \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha} \\ &\times\left[f(u_i(k)) - \kappa\sum_{j=1}^{M}\ell_{ij}Hu_j(k)\right], \qquad 1 \le n \le N, \end{aligned} \tag{4.4}$$

for $i = 1,\dots,M$, where $u_i(0) \in \mathbb{R}^d$ are prescribed initial states. Define

$$C_N := \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\max_{1\le n\le N}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha}.$$

If

$$\eta_N := C_N \left(\mathcal{L}_f + \kappa\lambda_M \|H\|_2\right) < 1, \tag{4.5}$$

then system (4.4) *has a unique mild solution on* $\mathbb{N}_N$. *Furthermore, define* $U(n) := \operatorname{col}\big(u_1(n), \ldots, u_M(n)\big) \in \mathbb{R}^{Md}$, *the orthogonal projection* $P := I_M - \frac{1}{M}\mathbf{1}_M\mathbf{1}_M^T$, *and the transverse disagreement vector* $E_\perp(n) := (P \otimes I_d)U(n)$. *Then*

$$\|E_\perp\|_{\infty,N} \le \frac{\|E_\perp(0)\|_2}{1-\eta_N}, \quad \|E_\perp\|_{\infty,N} := \max_{0\le n\le N} \|E_\perp(n)\|_2. \tag{4.6}$$

In particular, if $u_1(0) = u_2(0) = \cdots = u_M(0)$, *then* $u_1(n) = u_2(n) = \cdots = u_M(n)$ *for every* $n \in \mathbb{N}_N$. *Hence, the synchronization manifold is invariant on the finite time interval.*

Proof. Define the stacked nonlinear mapping $\mathcal{F} : \mathbb{R}^{Md} \to \mathbb{R}^{Md}$ by $\mathcal{F}(U) := \operatorname{col}\big(f(u_1), \ldots, f(u_M)\big)$. Let $U_0 := \operatorname{col}\big(u_1(0), \ldots, u_M(0)\big)$. Then system (4.4) can be written as

$$\begin{aligned} U(n) = U_0 + \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} &[(n-k)_{q,\tau}]^{-\alpha} \\ &\times \left[\mathcal{F}(U(k)) - \kappa(L_G \otimes H)U(k)\right], \qquad 1 \le n \le N. \end{aligned} \tag{4.7}$$

Consider the space $X_N^{Md} := \big\{U : \mathbb{N}_N \to \mathbb{R}^{Md}\big\}$ equipped with the norm $\|U\|_{\infty,N} := \max_{0\le n\le N} \|U(n)\|_2$. Since X_N^{Md} is finite-dimensional, it is a Banach space. Define $T : X_N^{Md} \to X_N^{Md}$ by $(TU)(0) := U_0$ and, for $1 \le n \le N$, by

$$\begin{aligned} (TU)(n) := U_0 + \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} &[(n-k)_{q,\tau}]^{-\alpha} \\ &\times \left[\mathcal{F}(U(k)) - \kappa(L_G \otimes H)U(k)\right]. \end{aligned}$$

Thus, all kernel terms appearing in T are positive and finite, so T is well defined. Let $U, V \in X_N^{Md}$. By the componentwise Lipschitz condition,

$$\begin{aligned} \|\mathcal{F}(U) - \mathcal{F}(V)\|_2^2 &= \sum_{i=1}^{M} \|f(u_i) - f(v_i)\|_2^2 \\ &\le \mathcal{L}_f^2 \sum_{i=1}^{M} \|u_i - v_i\|_2^2 \\ &= \mathcal{L}_f^2 \|U - V\|_2^2. \end{aligned}$$

Therefore,

$$\|\mathcal{F}(U) - \mathcal{F}(V)\|_2 \le \mathcal{L}_f \|U - V\|_2. \tag{4.8}$$

Since L_G is symmetric and positive semidefinite, $\|L_G\|_2 = \lambda_M$. Using the standard norm identity for Kronecker products, we obtain

$$\|L_G \otimes H\|_2 = \|L_G\|_2 \|H\|_2 = \lambda_M \|H\|_2.$$

Consequently,

$$\|(L_G \otimes H)(U - V)\|_2 \le \lambda_M \|H\|_2 \|U - V\|_2. \tag{4.9}$$

For $1 \le n \le N$, equations (4.8) and (4.9) give

$$\begin{aligned}\|(TU)(n)-(TV)(n)\|_2 &\le \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha}\\ &\quad\times\Big(\|\mathcal{F}(U(k))-\mathcal{F}(V(k))\|_2\\ &\quad+\kappa\|(L_G\otimes H)(U(k)-V(k))\|_2\Big)\\ &\le \frac{\mathcal{L}_f+\kappa\lambda_M\|H\|_2}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha}\|U(k)-V(k)\|_2\\ &\le \eta_N\|U-V\|_{\infty,N}.\end{aligned}$$

At $n=0$, $\|(TU)(0)-(TV)(0)\|_2=0$. Taking the maximum over $0\le n\le N$, we obtain $\|TU-TV\|_{\infty,N}\le\eta_N\|U-V\|_{\infty,N}$. Since $\eta_N<1$, the operator T is a contraction. Banach's fixed-point theorem therefore guarantees a unique fixed point $U\in X_N^{Md}$. This fixed point is the unique mild solution of (4.7), and hence of (4.4). We next establish the disagreement estimate. Let $J:=\frac{1}{M}\mathbf{1}_M\mathbf{1}_M^T$. Then $P=I_M-J$. Since $L_G\mathbf{1}_M=0$, the symmetry of L_G implies $PL_G=L_GP=L_G$. Therefore, $(P\otimes I_d)(L_G\otimes H)=L_G\otimes H$. Moreover, $(L_G\otimes H)U=(L_G\otimes H)E_\perp$, because $(L_G\otimes H)(J\otimes I_d)U=0$. Applying $P\otimes I_d$ to (4.7) gives

$$\begin{aligned}E_\perp(n)=E_\perp(0)&+\frac{1}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha}\\ &\times\left[(P\otimes I_d)\mathcal{F}(U(k))-\kappa(L_G\otimes H)E_\perp(k)\right].\end{aligned}$$

The vector $(J\otimes I_d)U$ belongs to the synchronization manifold; hence, all of its node components are identical. Since the same function f acts at every node, $\mathcal{F}\big((J\otimes I_d)U\big)$ also belongs to the synchronization manifold. It follows that $(P\otimes I_d)\mathcal{F}\big((J\otimes I_d)U\big)=0$. Thus,

$$\begin{aligned}\|(P\otimes I_d)\mathcal{F}(U)\|_2&=\left\|(P\otimes I_d)\left[\mathcal{F}(U)-\mathcal{F}\big((J\otimes I_d)U\big)\right]\right\|_2\\ &\le\left\|\mathcal{F}(U)-\mathcal{F}\big((J\otimes I_d)U\big)\right\|_2\\ &\le L_f\left\|U-(J\otimes I_d)U\right\|_2\\ &=L_f\|E_\perp\|_2.\end{aligned}$$

Here we used $\|P\otimes I_d\|_2=1$. Also, $\|(L_G\otimes H)E_\perp\|_2\le\lambda_M\|H\|_2\|E_\perp\|_2$. Taking norms in (4), using the preceding estimates, and then taking the maximum over $0\le n\le N$, we obtain $\|E_\perp\|_{\infty,N}\le\|E_\perp(0)\|_2+\eta_N\|E_\perp\|_{\infty,N}$. Because $\eta_N<1$, rearrangement gives $\|E_\perp\|_{\infty,N}\le\frac{\|E_\perp(0)\|_2}{1-\eta_N}$. This proves (4.6). Finally, suppose that $u_1(0)=u_2(0)=\cdots=u_M(0)$. Then U_0 belongs to the synchronization manifold, so $E_\perp(0)=0$. The disagreement estimate gives $\|E_\perp\|_{\infty,N}=0$. Therefore, $E_\perp(n)=0$ for every $n\in\mathbb{N}_N$, which is equivalent to $u_1(n)=u_2(n)=\cdots=u_M(n)$. Hence, the synchronization manifold is invariant on $\mathbb{N}_N$. □

Example 4.1 (Delayed three-node finite-horizon network)**.** Consider a network of three scalar oscillators governed by the delayed finite-horizon Volterra system

$$u_i(n)=u_i(0)+\frac{1}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha}$$

$$\times \left[\sin(u_i(k)) - \kappa \sum_{j=1}^{3} \ell_{ij} u_j(k-d)\right], \qquad n = 1, \ldots, N, \tag{4.10}$$

where the graph Laplacian is

$$L_G = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

The model parameters are selected as $\alpha = 0.85$, $q = 0.5$, $\tau = 1.2$, $q\tau = 0.6 < 1, \kappa = 0.5$, $d = 2$, $N = 50$. The prescribed history is constant on the delay interval: $u_1(n) = 0.1$, $u_2(n) = 0.2$, $u_3(n) = -0.1$, $-2 \le n \le 0$. For these parameters, numerical evaluation gives $\Gamma_{q,\tau}(1-\alpha) \approx 4.145998719498434$. For $n = 1, \ldots, N$, the solution is computed successively from

$$U_n = U_0 + \sum_{k=0}^{n-1} w_{n-k}^{(q,\tau,\alpha)} \left[\sin(U_k) - \kappa L_G U_{k-d}\right], \tag{4.11}$$

where the sine is evaluated componentwise and $w_r^{(q,\tau,\alpha)} = \frac{[(r)_{q,\tau}]^{-\alpha}}{\Gamma_{q,\tau}(1-\alpha)}$. When $k - d \le 0$, the prescribed history is used. Since every value U_n depends only on previously computed states, the recursion is explicit and does not require the solution of a nonlinear algebraic system at each step. Define the disagreement by

$$e_\perp(n) = \max_{1 \le i \le 3} |u_i(n) - \overline{u}(n)|, \qquad \overline{u}(n) = \frac{u_1(n) + u_2(n) + u_3(n)}{3}.$$

The results are:Use K_{ref}=200 as the reference calculation and define

Table 1. Computed states and transverse disagreement for the delayed three-node network.

n	$u_1(n)$	$u_2(n)$	$u_3(n)$	$e_\perp(n)$
0	0.100000	0.200000	-0.100000	0.166667
1	0.114530	0.199612	-0.056216	0.142191
2	0.128487	0.199241	-0.014125	0.118660
5	0.185998	0.223012	0.110228	0.062852
10	0.389001	0.418163	0.329574	0.049339
20	1.465443	1.470891	1.454249	0.009278
30	2.537888	2.538536	2.536553	0.001106
40	2.928334	2.928427	2.928143	0.000159
50	3.065716	3.065730	3.065687	0.000024

$$\mathcal{E}_K = \max_{0 \le n \le N} \|U^{[K]}(n) - U^{[200]}(n)\|_\infty.$$

The convergence results are: For two successive product truncations, define the computable a posteriori error indicator

$$\widehat{\mathcal{E}}_K = \max_{0 \le n \le N} \|U^{[2K]}(n) - U^{[K]}(n)\|_\infty.$$

Table 2. Convergence with respect to the truncation level K in the (q,τ)-Gamma product.

K	$\Gamma_{q,\tau}^{[K]}(1-\alpha)$	$\mathcal{E}_K$
5	4.094622224529	4.3481×10^{-2}
10	4.145196742377	6.7372×10^{-4}
20	4.145998523706	1.6446×10^{-7}
40	4.145998719498	9.9920×10^{-15}
80	4.145998719498	$< 10^{-15}$

The convergence table demonstrates rapid stabilization of both the deformed Gamma factor and the computed network trajectories as the product truncation level increases. At $K = 20$, the maximum deviation from the reference solution is below 2×10^{-7}, while at $K = 40$ the difference is at the level of double-precision rounding. The transverse disagreement decreases from approximately 0.1667 at the initial time to approximately 2.4×10^{-5} at the final time. Thus, the delayed trajectories become numerically synchronized over the selected finite interval. Table 1 shows that the three delayed node states gradually approach one another over the finite simulation interval. The transverse disagreement decreases from approximately 0.1667 at $n = 0$ to 2.4×10^{-5} at $n = 50$, indicating strong numerical synchronization despite the imposed delay. Table 2 confirms the numerical convergence of the scheme with respect to the product-truncation level. The error decreases rapidly as K increases, reaching approximately 1.6×10^{-7} at $K = 20$ and machine precision at $K = 40$. Thus, $K = 40$ provides sufficient numerical accuracy for the present simulation.

Theorem 4.2 (Synchronization with time-varying coupling and pinning control)**.** *Consider the (q,τ)-fractional network*

$$\nabla_{q,\tau}^{\alpha}\mathbf{u}_i(n) = \mathbf{f}(\mathbf{u}_i(n)) + \sum_{j=1}^{m} c_{ij}(n)H(\mathbf{u}_j(n) - \mathbf{u}_i(n)) - \delta_i B(\mathbf{u}_i(n) - \mathbf{s}(n)),$$

where $0 < \alpha < 1$, $0 < q\tau < 1$, H, B are symmetric positive definite, and $\mathbf{f}$ is globally Lipschitz with constant L. Assume that there exists $\mu > 0$ such that

$$\lambda_{\min}\big(\mathcal{L}(n) \otimes H + \mathcal{D} \otimes B\big) \geq \mu \quad \textit{for all } n,$$

and that $\mu > L$. Then the synchronization error $\mathbf{e}_i(n) = \mathbf{u}_i(n) - \mathbf{s}(n)$ is asymptotically stable. In particular, $\lim_{n\to\infty} \|\mathbf{u}_i(n) - \mathbf{s}(n)\| = 0$, $i = 1, \ldots, m$.

Proof. Let $\mathbf{e}_i(n) := \mathbf{u}_i(n) - \mathbf{s}(n)$, $\mathbf{e}(n) := \big(\mathbf{e}_1(n)^T, \ldots, \mathbf{e}_m(n)^T\big)^T \in \mathbb{R}^{md}$. Since $\mathbf{s}(n)$ satisfies $\nabla_{q,\tau}^{\alpha}\mathbf{s}(n) = \mathbf{f}(\mathbf{s}(n))$, subtracting this equation from the i-th node equation gives

$$\nabla_{q,\tau}^{\alpha}\mathbf{e}_i(n) = \mathbf{f}(\mathbf{u}_i(n)) - \mathbf{f}(\mathbf{s}(n)) + \sum_{j=1}^{m} c_{ij}(n)H\big(\mathbf{e}_j(n) - \mathbf{e}_i(n)\big) - \delta_i B\mathbf{e}_i(n).$$

Equivalently,

$$\nabla_{q,\tau}^{\alpha}\mathbf{e}(n) = \mathbf{F}(\mathbf{u}(n)) - \mathbf{F}(\mathbf{s}(n)) - \Big(\mathcal{L}(n) \otimes H + \mathcal{D} \otimes B\Big)\mathbf{e}(n),$$

where

$$\mathbf{F}(\mathbf{u}(n)) - \mathbf{F}(\mathbf{s}(n)) := \big(\mathbf{f}(\mathbf{u}_1(n)) - \mathbf{f}(\mathbf{s}(n)), \ldots, \mathbf{f}(\mathbf{u}_m(n)) - \mathbf{f}(\mathbf{s}(n))\big)^T,$$

$$\mathcal{D} = \operatorname{diag}(\delta_1, \ldots, \delta_m).$$

Define the Lyapunov function

$$V(n) := \frac{1}{2}\|\mathbf{e}(n)\|^2.$$

Using the standard fractional Lyapunov inequality for the (q,τ)-nabla operator,

$$\nabla^{\alpha}_{q,\tau} V(n) \le \mathbf{e}(n)^T \nabla^{\alpha}_{q,\tau} \mathbf{e}(n),$$

we obtain

$$\nabla^{\alpha}_{q,\tau} V(n) \le \mathbf{e}(n)^T \big(\mathbf{F}(\mathbf{u}(n)) - \mathbf{F}(\mathbf{s}(n))\big) - \mathbf{e}(n)^T \Big(\mathcal{L}(n) \otimes H + \mathcal{D} \otimes B\Big)\mathbf{e}(n).$$

Since $\mathbf{f}$ is globally Lipschitz with constant L, we have

$$\|\mathbf{f}(\mathbf{u}_i(n)) - \mathbf{f}(\mathbf{s}(n))\| \le L\|\mathbf{e}_i(n)\|.$$

Hence,

$$\mathbf{e}(n)^T \big(\mathbf{F}(\mathbf{u}(n)) - \mathbf{F}(\mathbf{s}(n))\big) \le L\|\mathbf{e}(n)\|^2.$$

Moreover, by assumption, there exists $\mu > 0$ such that

$$\lambda_{\min}\Big(\mathcal{L}(n) \otimes H + \mathcal{D} \otimes B\Big) \ge \mu \qquad \text{for all } n.$$

Therefore,

$$\mathbf{e}(n)^T \Big(\mathcal{L}(n) \otimes H + \mathcal{D} \otimes B\Big)\mathbf{e}(n) \ge \mu\|\mathbf{e}(n)\|^2.$$

Combining the above estimates yields

$$\nabla^{\alpha}_{q,\tau} V(n) \le L\|\mathbf{e}(n)\|^2 - \mu\|\mathbf{e}(n)\|^2.$$

Thus, $\nabla^{\alpha}_{q,\tau} V(n) \le -(\mu - L)\|\mathbf{e}(n)\|^2$. Since $\mu > L$, let $\eta := \mu - L > 0$. Then $\nabla^{\alpha}_{q,\tau} V(n) \le -\eta\|\mathbf{e}(n)\|^2 = -2\eta V(n)$. That is, $\nabla^{\alpha}_{q,\tau} V(n) \le -2\eta V(n)$. Consequently, $\lim_{n\to\infty} \|\mathbf{e}(n)\| = 0$. Since $\|\mathbf{e}_i(n)\| \le \|\mathbf{e}(n)\|$, we conclude that $\lim_{n\to\infty} \|\mathbf{u}_i(n) - \mathbf{s}(n)\| = 0, \qquad i = 1, \ldots, m$. Therefore, the delayed (q,τ)-fractional network achieves global asymptotic synchronization. □

Example 4.2 (Time-varying coupling and pinning control)**.** Consider three scalar nodes by

$$\nabla^{\alpha}_{q,\tau} u_i(n) = -\sin(u_i(n)) - \kappa \sum_{j=1}^{3} \ell_{ij}(n) u_j(n) - \delta_i B\big(u_i(n) - s(n)\big), \tag{4.12}$$

where $\alpha = 0.8$, $q = 0.5$, $\tau = 1.2$, $q\tau = 0.6 < 1, \kappa = 0.5$, $B = 1$, $\delta = (1,0,0)^T$, $s(n) \equiv 0$. The initial state is $U(0) = (0.1, 0.2, -0.1)^T$. The time-varying positive-semidefinite graph Laplacian is

$$L_G(n) = \begin{cases} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, & n \text{ even}, \\ \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, & n \text{ odd}. \end{cases}$$

Define $D = \mathrm{diag}(1, 0, 0)$ and $G(U(k), k) = -\sin(U(k)) - \kappa L_G(k)U(k) - BDU(k)$, where the sine is evaluated componentwise. The numerical solution is computed explicitly from

$$U(n) = U(0) + \sum_{k=0}^{n-1} w_{n-k}^{(\alpha,q,\tau)} G(U(k), k), \qquad n = 1, \ldots, N, \tag{4.13}$$

where $w_r^{(\alpha,q,\tau)} = \frac{[(r)_{q,\tau}]^{-\alpha}}{\Gamma_{q,\tau}(1-\alpha)}$. Because the model is intrinsically discrete, no time-step refinement is required. On the finite horizon, the memory sum is evaluated completely. The only truncation is in the infinite product used to calculate the deformed Gamma function. Convergence criterion is as follows: let $\Gamma_{q,\tau}^{[K]}$ denote the approximation obtained from the first $[K]$ factors of the defining product, and let $U^{[K]}$ be the associated numerical solution. Define

$$\mathcal{E}_K = \max_{0 \le n \le N} \|U^{[2K]}(n) - U^{[K]}(n)\|_\infty.$$

The computation is declared converged when $\mathcal{E}_K < 10^{-8}$. The residual may also be monitored using

$$\mathcal{R}_K = \max_{1 \le n \le N} \left\| U^{[K]}(n) - U(0) - \sum_{k=0}^{n-1} w_{n-k}^{[K]} G(U^{[K]}(k), k) \right\|_\infty .$$

For the explicit recursion, this residual is at the level of floating-point roundoff. Truncation and error analysis is occurred using $K_{ref} = 200$ as the reference computation, with $N = 50$. The calculated results are:Thus, the convergence tolerance is reached at K=20, while K=40 gives

Table 3. Convergence and truncation error for Example 2.

K	$\Gamma_{q,\tau}^{[K]}(1-\alpha)$	E_K
5	3.124166057433	1.7390×10^{-3}
10	3.159628182932	2.6832×10^{-5}
20	3.160190210208	6.5495×10^{-9}
40	3.160190347453	4.1633×10^{-16}
80	3.160190347453	$< 10^{-15}$

essentially machine-precision accuracy. Recalculated solution, we obtain Table 3 demonstrates rapid convergence with respect to the product-truncation level. The maximum error falls below 10^{-8} at $K = 20$ and reaches machine precision at $K = 40$. Table 4 shows that all three node states approach the pinned reference over the simulated interval, despite the alternating graph topology. This is finite-horizon numerical evidence and is not interpreted as an independent proof of asymptotic synchronization.

4.1. Numerical assessment of the sufficient boundary

To assess the practical meaning of the sufficient finite-horizon condition, we consider parameter regimes below, near, and above its theoretical boundary. Because the condition is sufficient rather than necessary, a value above the boundary is described as *unsupported*, not necessarily unstable. We use the time-varying three-node network

$$U(n) = U(0) + \sum_{k=0}^{n-1} w_{n-k}^{(\alpha,q,\tau)} \left[-a \sin(U(k)) - \kappa L_G(k)U(k) - D_B U(k) \right], \tag{4.14}$$

Table 4. Recalculated trajectories for the time-varying pinned network.

n	$u_1(n)$	$u_2(n)$	$u_3(n)$	$\max_i \lvert u_i(n)\rvert$
0	1.0000×10^{-1}	2.0000×10^{-1}	-1.0000×10^{-1}	2.0000×10^{-1}
1	4.3321×10^{-2}	4.9191×10^{-2}	-5.4927×10^{-3}	4.9191×10^{-2}
5	2.1682×10^{-2}	3.0376×10^{-2}	-5.0849×10^{-3}	3.0376×10^{-2}
10	4.5360×10^{-3}	1.3609×10^{-2}	1.9083×10^{-3}	1.3609×10^{-2}
20	1.0922×10^{-3}	2.5617×10^{-3}	9.6333×10^{-4}	2.5617×10^{-3}
30	2.5952×10^{-4}	5.1385×10^{-4}	2.9540×10^{-4}	5.1385×10^{-4}
40	6.1299×10^{-5}	1.0828×10^{-4}	7.8499×10^{-5}	1.0828×10^{-4}
50	1.4411×10^{-5}	2.3651×10^{-5}	1.9606×10^{-5}	2.3651×10^{-5}

where the sine is evaluated componentwise, $D_B = \mathrm{diag}(B, 0, 0)$, and

$$L_G(k) = \begin{cases} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, & k \text{ even}, \\ \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, & k \text{ odd}. \end{cases}$$

The kernel weights are $w_r^{(\alpha,q,\tau)} = \frac{[(r)_{q,\tau}]^{-\alpha}}{\Gamma_{q,\tau}(1-\alpha)}$. The parameters held fixed in the three simulations are $\alpha = 0.8$, $q = 0.5$, $\tau = 1.2$, $N = 50, a = 0.04$, $B = 0.02$, $U(0) = (0.1, 0.2, -0.1)^T$. For these parameters, $q\tau = 0.6 < 1$, and numerical evaluation gives

$$C_N^{(\alpha,q,\tau)} = \max_{1 \le n \le N} \sum_{k=0}^{n-1} w_{n-k}^{(\alpha,q,\tau)} \approx 9.451759689.$$

The largest spectral norm of the two graph Laplacians is $\max_k \|L_G(k)\|_2 = 3$. Since the Lipschitz constant of the local response $-a\sin(x)$ is a, a sufficient contraction indicator for this example is

$$\eta_N = C_N^{(\alpha,q,\tau)} (a + 3\kappa + B), \quad \eta_N < 1. \tag{4.15}$$

Three coupling values were selected to obtain a supported regime, a near-threshold regime, and an unsupported regime:

$$\kappa_{\mathrm{s}} = 0.00821344, \qquad \kappa_{\mathrm{nt}} = 0.01456147, \qquad \kappa_{\mathrm{u}} = 0.02232016.$$

The corresponding indicator values are approximately 0.80, 0.98, and 1.20, respectively. The transverse disagreement is measured by

$$e_\perp(n) = \max_{1 \le i \le 3} |u_i(n) - \overline{u}(n)|, \qquad \overline{u}(n) = \frac{1}{3} \sum_{i=1}^{3} u_i(n). \tag{4.16}$$

We also calculate the maximum recursion residual

$$\mathcal{R} = \max_{1\le n\le N} \left\| U(n) - U(0) - \sum_{k=0}^{n-1} w_{n-k}^{(\alpha,q,\tau)} G(U(k),k) \right\|_{\infty}, \tag{4.17}$$

where $G(U(k),k) = -a\sin(U(k)) - \kappa L_G(k)U(k) - D_B U(k)$. Table 5 shows that the supported

Table 5. Comparison of supported, near-threshold, and unsupported parameter regimes.

Regime	κ	η_N	$e_\perp(0)$	$e_\perp(50)$	$e_\perp(50)/e_\perp(0)$	$\mathcal{R}$
Supported	0.008213	0.80	0.166667	0.092005	0.5520	8.33×10^{-17}
Near-threshold	0.014561	0.98	0.166667	0.079807	0.4788	9.71×10^{-17}
Unsupported	0.022320	1.20	0.166667	0.067214	0.4033	9.71×10^{-17}

regime satisfies the sufficient contraction condition and exhibits bounded finite-horizon evolution with decreasing disagreement. In the near-threshold regime, the theoretical margin is small, so the analytical contraction bound becomes conservative, although the computed disagreement still decreases. The unsupported regime also shows decreasing disagreement in this particular simulation. This does not contradict the theorem because failure of a sufficient condition does not imply instability. It means only that the contraction proof no longer guarantees the observed behavior. In this experiment, increasing the diffusive coupling improves the numerical disagreement even though the norm-based contraction indicator increases, because the norm estimate treats the coupling term through the triangle inequality and therefore does not exploit its negative dissipative sign. These results demonstrate that the sufficient boundary correctly identifies a rigorously supported parameter region but is conservative as a predictor of the actual synchronization transition. Accordingly, the theoretical condition is interpreted as a guarantee rather than a necessary stability boundary.

5. Exact energy identity for the (q,τ)-fractional network

This section derives an exact finite-horizon identity for the (q,τ)-fractional action on the quadratic synchronization energy. Because the adopted operator is nonlocal, the classical quadratic chain rule cannot generally be applied by replacing the fractional energy difference with an inner product evaluated only at the current time. Instead, the correct expression involves the complete collection of weighted state increments over the preceding time interval. The resulting identity separates the energy variation into a memory-weighted inner-product term and an explicit nonpositive quadratic remainder. This formulation avoids introducing an unspecified constant and makes the dependence on the fractional order and deformation parameters explicit through the kernel weights. The identity established below is algebraic and exact on every finite time horizon. By itself, it does not prove asymptotic synchronization. Any subsequent stability conclusion requires additional dissipativity conditions and a comparison principle proved for the particular (q,τ)-fractional kernel.

Proposition 5.1 (Exact finite-horizon fractional energy identity)**.** *Let* $E(n) := \sum_{i=1}^{m} \|\mathbf{e}_i(n)\|_2^2$ *and define* $\Delta\mathbf{e}_i(k) := \mathbf{e}_i(k) - \mathbf{e}_i(k+1)$. *Then*

$$\nabla_{q,\tau}^{\alpha} E(n) = \frac{2}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} \frac{\sum_{i=1}^{m} \langle \mathbf{e}_i(k), \Delta\mathbf{e}_i(k)\rangle}{[(n-k)_{q,\tau}]^{\alpha}} - \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} \frac{\sum_{i=1}^{m} \|\Delta\mathbf{e}_i(k)\|_2^2}{[(n-k)_{q,\tau}]^{\alpha}}. \tag{5.1}$$

Consequently,

$$\nabla^{\alpha}_{q,\tau}E(n) \le \frac{2}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}\frac{\sum_{i=1}^{m}\langle \mathbf{e}_i(k), \Delta\mathbf{e}_i(k)\rangle}{[(n-k)_{q,\tau}]^{\alpha}}. \tag{5.2}$$

Proof. By the linearity of $\nabla^{\alpha}_{q,\tau}$, we have

$$\nabla^{\alpha}_{q,\tau}E(n) = \sum_{i=1}^{m}\nabla^{\alpha}_{q,\tau}\|\mathbf{e}_i(n)\|_2^2.$$

Using the definition of the fractional nabla operator gives

$$\nabla^{\alpha}_{q,\tau}\|\mathbf{e}_i(n)\|_2^2 = \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}\frac{\|\mathbf{e}_i(k)\|_2^2 - \|\mathbf{e}_i(k+1)\|_2^2}{[(n-k)_{q,\tau}]^{\alpha}}.$$

Define $\Delta\mathbf{e}_i(k) := \mathbf{e}_i(k) - \mathbf{e}_i(k+1)$. The elementary identity

$$\|a\|_2^2 - \|b\|_2^2 = 2\langle a, a-b\rangle - \|a-b\|_2^2$$

with $a = \mathbf{e}_i(k), \qquad b = \mathbf{e}_i(k+1)$ yields

$$\|\mathbf{e}_i(k)\|_2^2 - \|\mathbf{e}_i(k+1)\|_2^2 = 2\,\langle \mathbf{e}_i(k), \Delta\mathbf{e}_i(k)\rangle - \|\Delta\mathbf{e}_i(k)\|_2^2.$$

Substituting this identity into the definition of the fractional operator and summing over $i = 1, \ldots, m$, we obtain

$$\nabla^{\alpha}_{q,\tau}E(n) = \frac{2}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}\frac{\sum_{i=1}^{m}\langle \mathbf{e}_i(k), \Delta\mathbf{e}_i(k)\rangle}{[(n-k)_{q,\tau}]^{\alpha}} - \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}\frac{\sum_{i=1}^{m}\|\Delta\mathbf{e}_i(k)\|_2^2}{[(n-k)_{q,\tau}]^{\alpha}}.$$

Because $0 < q\tau < 1$ and $\Gamma_{q,\tau}(1-\alpha) > 0$, all kernel weights are positive. Therefore, the second sum is nonnegative and appears with a negative sign. Dropping this nonpositive term gives

$$\nabla^{\alpha}_{q,\tau}E(n) \le \frac{2}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}\frac{\sum_{i=1}^{m}\langle \mathbf{e}_i(k), \Delta\mathbf{e}_i(k)\rangle}{[(n-k)_{q,\tau}]^{\alpha}}.$$

This proves the exact identity and its upper bound. □

5.1. Application to resilient event-triggered networks

Resilient event-triggered networks provide an important application of the proposed (q,τ)-fractional energy identity. In such systems, agents do not exchange information continuously. Instead, communication is activated only when a local error measure exceeds a prescribed threshold. This reduces communication cost while maintaining stability and synchronization. Let $\xi_i(n) = \|\mathbf{e}_i(n) - \mathbf{e}_i(n_i^{\ell})\|$, where n_i^{ℓ} denotes the last triggering instant of node i. A typical event-triggering rule is given by

$$\xi_i(n) \ge \sigma_i(n)\|\mathbf{e}_i(n)\|, \qquad 0 < \sigma_i(n) < 1.$$

When this condition is satisfied, node i broadcasts its state to its neighbors; otherwise, the previous transmitted state is used. The corresponding resilient network can be written as

$$\nabla^{\alpha}_{q,\tau}\mathbf{u}_i(n) = \mathbf{f}(\mathbf{u}_i(n)) + \sum_{j=1}^{m}a_{ij}H\big(\widehat{\mathbf{u}}_j(n) - \widehat{\mathbf{u}}_i(n)\big) + \mathbf{d}_i(n),$$

where $\widehat{\mathbf{u}}_i(n)$ is the most recently transmitted state and $\mathbf{d}_i(n)$ represents disturbances, packet losses, or delay effects. The role of the (q,τ)-fractional nabla operator is essential in this setting. Because communication is intermittent, the current state may not contain sufficient information to guarantee convergence. The memory kernel in $\nabla^{\alpha}_{q,\tau}$ incorporates previous states and therefore compensates for missing or delayed updates. Using the energy $E(n) = \sum_{i=1}^{m} \|\mathbf{e}_i(n)\|^2$, the discrete (q,τ)-fractional energy identity gives

$$\nabla^{\alpha}_{q,\tau} E(n) \leq \frac{2}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} \frac{1}{[(n-k)_{(q,\tau)}]^{\alpha}} \sum_{i=1}^{m} \langle \mathbf{e}_i(k), \mathbf{e}_i(k) - \mathbf{e}_i(k+1) \rangle .$$

This inequality provides a memory-weighted dissipation estimate for the global synchronization error. If the triggering threshold is chosen so that the communication error remains dominated by the dissipative coupling term, then $\nabla^{\alpha}_{q,\tau} E(n) \leq -\kappa E(n)$, $\kappa > 0$. Hence, by the discrete (q,τ)-fractional Lyapunov stability theorem, $E(n) \to 0$ as $n \to \infty$. Therefore, the event-triggered network achieves synchronization while reducing the number of transmissions. This framework is useful for cyber–physical systems, wireless sensor networks, robotic swarms, smart grids, and distributed control systems, where continuous communication is expensive or unreliable. The parameters q and τ provide additional flexibility: q controls the deformation of the memory weights, while τ adjusts the temporal scaling of past information. Thus, the (q,τ)-fractional formulation improves robustness against delays, packet losses, and sparse communication. In

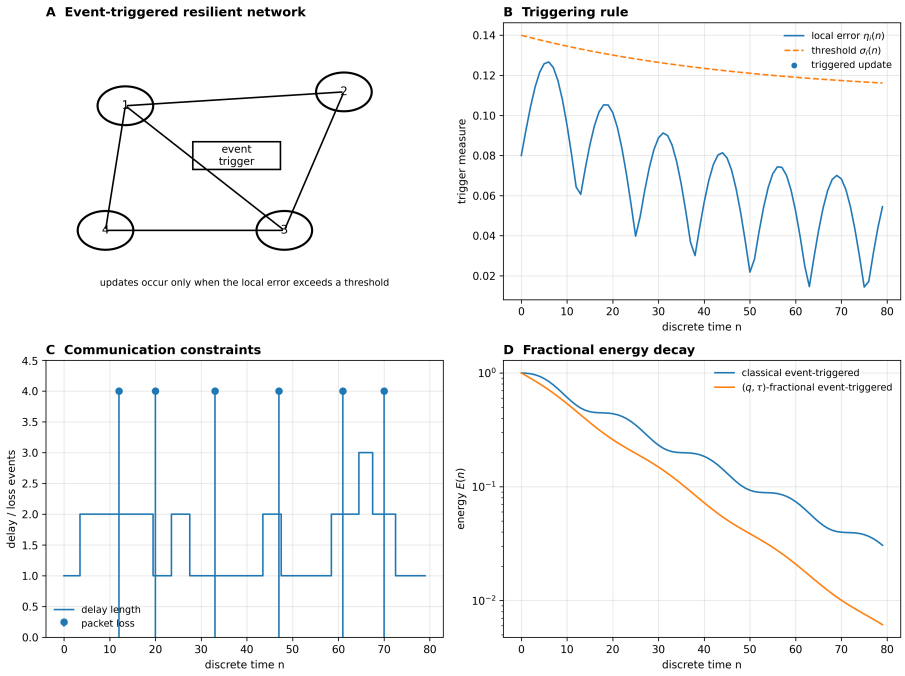

Figure 2. Resilient event-triggered network under (q,τ)-fractional memory. (A) Event-triggered network architecture with intermittent communication. (B) Triggering rule showing local error and threshold crossing. (C) Communication constraints caused by delays and packet losses. (D) Decay of the synchronization energy, showing improved convergence under the (q,τ)-fractional event-triggered strategy.

Figure 2, Panel (A) shows the event-triggered network structure, where communication occurs only when needed. Panel (B) illustrates the triggering mechanism, with fewer events over

time as the system approaches synchronization. Panel (C) demonstrates robustness under delays and packet losses. Panel (D) shows the decay of the synchronization energy, where the (q,τ)-fractional model achieves faster and smoother convergence compared to the classical case. Overall, the figure highlights improved efficiency and resilience due to fractional memory.

Theorem 5.1 (Finite-horizon synchronization-energy decay under an incremental dissipation condition)**.** *Let* $E(n) := \sum_{i=1}^{m} \|\mathbf{e}_i(n)\|_2^2$, $n = 0, \dots, N$, *and suppose that* $0 < \alpha < 1$, $0 < q < 1$, $\tau > 0$, $0 < q\tau < 1$, *with* $\Gamma_{q,\tau}(1-\alpha) > 0$. *Define* $\Delta\mathbf{e}_i(k) := \mathbf{e}_i(k) - \mathbf{e}_i(k+1)$. *Assume that there exists a constant* $\vartheta = \vartheta(\kappa, L_G, H, \mu) \in (0,1)$ *such that, for every* $k = 0, \dots, N-1$,

$$2\sum_{i=1}^{m} \langle \mathbf{e}_i(k), \Delta\mathbf{e}_i(k) \rangle - \sum_{i=1}^{m} \|\Delta\mathbf{e}_i(k)\|_2^2 \geq \vartheta E(k). \tag{5.3}$$

Then

$$E(k+1) \leq (1-\vartheta)E(k), \qquad k = 0, \dots, N-1, \tag{5.4}$$

and consequently

$$E(n) \leq (1-\vartheta)^n E(0), \qquad n = 0, \dots, N. \tag{5.5}$$

Moreover,

$$\nabla_{q,\tau}^{\alpha} E(n) \geq \frac{\vartheta}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} \frac{E(k)}{[(n-k)_{q,\tau}]^{\alpha}} \geq 0, \qquad 1 \leq n \leq N. \tag{5.6}$$

Thus, the finite-horizon energy decay is represented by a positive memory-weighted dissipation quantity depending explicitly on α, q, τ.

Proof. The elementary identity

$$E(k) - E(k+1) = 2\sum_{i=1}^{m} \langle \mathbf{e}_i(k), \Delta\mathbf{e}_i(k) \rangle - \sum_{i=1}^{m} \|\Delta\mathbf{e}_i(k)\|_2^2$$

and assumption (5.3) give $E(k) - E(k+1) \geq \vartheta E(k)$. Therefore, $E(k+1) \leq (1-\vartheta)E(k)$. Iteration yields $E(n) \leq (1-\vartheta)^n E(0)$. By the definition of the fractional nabla operator,

$$\nabla_{q,\tau}^{\alpha} E(n) = \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} \frac{E(k) - E(k+1)}{[(n-k)_{q,\tau}]^{\alpha}}.$$

All kernel weights are positive under the stated assumptions. Substituting $E(k) - E(k+1) \geq \vartheta E(k)$ gives

$$\nabla_{q,\tau}^{\alpha} E(n) \geq \frac{\vartheta}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} \frac{E(k)}{[(n-k)_{q,\tau}]^{\alpha}} \geq 0.$$

□

Remark 5.1 (Roles of the decay and fractional parameters)**.** The constant $\vartheta \in (0,1)$ denotes an assumed one-step geometric energy-decay margin and is not, under the present assumptions, an explicitly derived function of α, q, or τ. The parameters α, q, and τ instead determine the memory-weighted accumulation of the energy decrements through the deformed kernel and the corresponding Gamma factor. Likewise, the coupling strength κ, graph Laplacian L_G, inner-coupling matrix H, and nonlinear dissipativity constant μ characterize the instantaneous

transverse coercivity of the network vector field, but they do not yield an explicit formula for ϑ without an additional one-step evolution law, resolvent representation, or comparison principle for the adopted (q,τ)-fractional operator.

Theorem 5.2 (Spectral-set invariance and energy dissipation under adaptive pinning)**.** *Let $M \geq 2$ be the number of nodes and $d \geq 1$ the dimension of each node state. Consider the adaptively pinned network*

$$\nabla_{q,\tau}^{\alpha} u_i(n) = f(u_i(n)) - \kappa \sum_{j=1}^{M} \ell_{ij} H u_j(n) - \delta_i B_i(n)\big(u_i(n) - s(n)\big), \qquad i = 1, \ldots, M, \tag{5.7}$$

where $0 < \alpha < 1$, $0 < q < 1$, $\tau > 0$, $0 < q\tau < 1$, and $\kappa > 0$. Assume the following:

(A1) *$L_G = (\ell_{ij}) \in \mathbb{R}^{M \times M}$ is the fixed symmetric positive-semidefinite Laplacian matrix of a connected undirected graph.*

(A2) *$H \in \mathbb{R}^{d \times d}$ is symmetric positive definite.*

(A3) *The reference trajectory $s(n) \in \mathbb{R}^d$ satisfies $\nabla_{q,\tau}^{\alpha} s(n) = f(s(n))$.*

(A4) *The function $f : \mathbb{R}^d \to \mathbb{R}^d$ satisfies the one-sided Lipschitz condition*

$$\langle x - y, f(x) - f(y) \rangle \leq \mathcal{L}_f \|x - y\|_2^2, \qquad x, y \in \mathbb{R}^d, \tag{5.8}$$

for some $\mathcal{L}_f \geq 0$.

(A5) *The pinning indicators satisfy $\delta_i \in \{0, 1\}$, $\sum_{i=1}^{M} \delta_i \geq 1$.*

(A6) *For each pinned node, the adaptive gain satisfies*

$$B_i(n+1) = B_i(n) + \gamma_i \|u_i(n) - s(n)\|_2^2, \qquad B_i(0) > 0, \quad \gamma_i > 0. \tag{5.9}$$

(A7) *The adopted (q,τ)-fractional operator satisfies the quadratic energy inequality*

$$\nabla_{q,\tau}^{\alpha} \left(\frac{1}{2} \|e(n)\|_2^2 \right) \leq \big\langle e(n), \nabla_{q,\tau}^{\alpha} e(n) \big\rangle \tag{5.10}$$

for the error sequence considered below.

Define $e_i(n) := u_i(n) - s(n)$ and $e(n) := \operatorname{col}\big(e_1(n), \ldots, e_M(n)\big) \in \mathbb{R}^{Md}$. Let

$$D_B(n) := \operatorname{diag}\big(\delta_1 B_1(n), \ldots, \delta_M B_M(n)\big)$$

and define the complete controlled network matrix

$$\mathcal{Q}(n) := \kappa L_G \otimes H + D_B(n) \otimes I_d. \tag{5.11}$$

Suppose that there exist a finite integer $n_0 \geq 0$ and a constant $\varepsilon > 0$ such that

$$\lambda_{\min}\big(\mathcal{Q}(n_0)\big) \geq \mathcal{L}_f + \varepsilon. \tag{5.12}$$

Then the spectral stability condition is forward invariant:

$$\lambda_{\min}\big(\mathcal{Q}(n)\big) \geq \mathcal{L}_f + \varepsilon, \qquad n \geq n_0. \tag{5.13}$$

Moreover, the synchronization energy $V(n) := \frac{1}{2}\|e(n)\|_2^2$ satisfies

$$\nabla_{q,\tau}^{\alpha} V(n) \leq -2\varepsilon V(n), \qquad n \geq n_0. \tag{5.14}$$

Proof. We first prove the forward invariance of the spectral condition. For every pinned node, the adaptive law (5.9) gives $B_i(n+1) - B_i(n) = \gamma_i \|e_i(n)\|_2^2 \geq 0$. Thus, every pinned gain is nondecreasing. For an unpinned node, $\delta_i = 0$, so its gain does not contribute to the pinning matrix. Therefore,

$$D_B(n+1) - D_B(n) = \operatorname{diag}\Big(\delta_1\gamma_1\|e_1(n)\|_2^2, \ldots, \delta_M\gamma_M\|e_M(n)\|_2^2\Big) \succeq 0.$$

It follows by induction that

$$D_B(n) \succeq D_B(n_0), \qquad n \geq n_0. \tag{5.15}$$

Since L_G and H are fixed, the difference between the controlled matrices at times n and n_0 is

$$\begin{aligned}\mathcal{Q}(n) - \mathcal{Q}(n_0) &= [\kappa L_G \otimes H + D_B(n) \otimes I_d] - [\kappa L_G \otimes H + D_B(n_0) \otimes I_d] \\ &= \big(D_B(n) - D_B(n_0)\big) \otimes I_d.\end{aligned}$$

By (5.15), $\mathcal{Q}(n) - \mathcal{Q}(n_0) \succeq 0$. Hence, $\mathcal{Q}(n) \succeq \mathcal{Q}(n_0)$, $n \geq n_0$. The minimum eigenvalue is monotone with respect to the Loewner order. Consequently,

$$\lambda_{\min}\big(\mathcal{Q}(n)\big) \geq \lambda_{\min}\big(\mathcal{Q}(n_0)\big) \geq \mathcal{L}_f + \varepsilon.$$

This proves (5.13). We now derive the energy-dissipation estimate. Since $L_G\mathbf{1}_M = 0$, the coupling term vanishes on the synchronized reference trajectory. Subtracting $\nabla_{q,\tau}^{\alpha} s(n) = f(s(n))$ from each node equation gives

$$\nabla_{q,\tau}^{\alpha} e_i(n) = f(s(n) + e_i(n)) - f(s(n)) - \kappa \sum_{j=1}^{M} \ell_{ij} H e_j(n) - \delta_i B_i(n) e_i(n).$$

In stacked form, this becomes

$$\nabla_{q,\tau}^{\alpha} e(n) = \mathcal{G}(e(n), s(n)) - \mathcal{Q}(n) e(n), \tag{5.16}$$

where

$$\mathcal{G}(e, s) := \operatorname{col}\big(f(s + e_1) - f(s), \ldots, f(s + e_M) - f(s)\big).$$

Using the one-sided Lipschitz condition (5.8), we obtain

$$\begin{aligned}\langle e(n), \mathcal{G}(e(n), s(n))\rangle &= \sum_{i=1}^{M} \langle e_i(n), f(s(n) + e_i(n)) - f(s(n))\rangle \\ &\leq L_f \sum_{i=1}^{M} \|e_i(n)\|_2^2 \\ &= L_f \|e(n)\|_2^2.\end{aligned}$$

Furthermore, the spectral estimate (5.13) implies

$$\langle e(n), \mathcal{Q}(n) e(n)\rangle \geq \lambda_{\min}\big(\mathcal{Q}(n)\big)\|e(n)\|_2^2 \geq (\mathcal{L}_f + \varepsilon)\|e(n)\|_2^2.$$

Taking the inner product of (5.16) with $e(n)$, we obtain

$$\begin{aligned}\big\langle e(n), \nabla_{q,\tau}^{\alpha} e(n)\big\rangle &= \langle e(n), \mathcal{G}(e(n), s(n))\rangle - \langle e(n), \mathcal{Q}(n) e(n)\rangle \\ &\leq \mathcal{L}_f \|e(n)\|_2^2 - (\mathcal{L}_f + \varepsilon)\|e(n)\|_2^2 \\ &= -\varepsilon \|e(n)\|_2^2.\end{aligned}$$

By the quadratic energy inequality (5.10),

$$\begin{aligned}\nabla^{\alpha}_{q,\tau}V(n) &= \nabla^{\alpha}_{q,\tau}\left(\frac{1}{2}\|e(n)\|_2^2\right)\\ &\le \left\langle e(n), \nabla^{\alpha}_{q,\tau}e(n)\right\rangle\\ &\le -\varepsilon\|e(n)\|_2^2.\end{aligned}$$

Since $\|e(n)\|_2^2 = 2V(n)$, we conclude that $\nabla^{\alpha}_{q,\tau}V(n) \le -2\varepsilon V(n)$, $n \ge n_0$. This proves (5.14). Finally, iteration of the adaptive law gives $B_i(n) = B_i(0) + \gamma_i \sum_{k=0}^{n-1}\|e_i(k)\|_2^2$ for each pinned node. This formula shows that $B_i(n)$ is nondecreasing. It does not imply that $B_i(n) \to \infty$. In particular, if $\sum_{k=0}^{\infty}\|e_i(k)\|_2^2 < \infty$, then $B_i(n)$ converges to a finite limit. Therefore, no unbounded-gain conclusion is used. If a valid comparison principle for the adopted fractional operator establishes that $\nabla^{\alpha}_{q,\tau}V(n) \le -2\varepsilon V(n)$ implies $V(n) \to 0$, then $\|e(n)\|_2 \to 0$. Since $\|e_i(n)\|_2 \le \|e(n)\|_2$, we obtain $\|u_i(n) - s(n)\|_2 = \|e_i(n)\|_2 \to 0$ for every $i = 1, \dots, M$. □

Theorem 5.3 (Finite-horizon well-posedness and continuous dependence for delayed (q,τ)-Volterra systems)**.** *Let $0 < \alpha < 1$, $0 < q < 1$, $\tau > 0$, $0 < q\tau < 1$, and let $d \in \mathbb{N}$, $N \in \mathbb{N}$. Define $\mathbb{N}_N := \{0, 1, \dots, N\}$ and $\mathbb{N}_{-d,0} := \{-d, -d+1, \dots, 0\}$, $\Gamma_{q,\tau}(1-\alpha) > 0$. Let $f : \mathbb{N}_{N-1} \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ satisfy*

$$\|f(n, x_1, y_1) - f(n, x_2, y_2)\|_2 \le \mathcal{L}_1\|x_1 - x_2\|_2 + \mathcal{L}_2\|y_1 - y_2\|_2 \tag{5.17}$$

for every $n \in \mathbb{N}_{N-1}$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}^m$, where $\mathcal{L}_1, \mathcal{L}_2 > 0$. Let $\phi : \mathbb{N}_{-d,0} \longrightarrow \mathbb{R}^m$ be a prescribed initial history. For a sequence $u : \mathbb{N}_N \to \mathbb{R}^m$, define its history extension by

$$\widehat{u}_{\phi}(r) := \begin{cases}\phi(r), & -d \le r \le 0,\\ u(r), & 1 \le r \le N.\end{cases} \tag{5.18}$$

Consider the delayed finite-horizon Volterra equation

$$u(n) = \phi(0) + \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha} \times f\left(k, u(k), \widehat{u}_{\phi}(k-d)\right),\ 1 \le n \le N, \tag{5.19}$$

with $u(0) = \phi(0)$. Define

$$C_N := \frac{1}{\Gamma_{q,\tau}(1-\alpha)}\max_{1\le n\le N}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha}. \tag{5.20}$$

If

$$(\mathcal{L}_1 + \mathcal{L}_2)C_N < 1, \tag{5.21}$$

then equation (5.19) has a unique mild solution $u : \mathbb{N}_N \longrightarrow \mathbb{R}^m$. Moreover, let u and v be the mild solutions corresponding to the histories ϕ and ψ, respectively. Define $\|\phi - \psi\|_{\infty,-d,0} := \max_{-d\le r\le 0}\|\phi(r) - \psi(r)\|_2$. Then

$$\|u - v\|_{\infty,N} \le \frac{\|\phi - \psi\|_{\infty,-d,0}}{1 - (L_1 + L_2)C_N}, \tag{5.22}$$

where $\|u - v\|_{\infty,N} := \max_{0\le n\le N}\|u(n) - v(n)\|_2$. If, in addition,

$$f(n, 0, 0) = 0, \qquad n \in \mathbb{N}_{N-1}, \tag{5.23}$$

then the zero history $\phi(r) = 0$, $-d \le r \le 0$, generates the unique zero mild solution $u(n) = 0$, $0 \le n \le N$.

Proof. Let $X_N^m := \{u : \mathbb{N}_N \to \mathbb{R}^m\}$ be equipped with the norm $\|u\|_{\infty,N} := \max_{0\le n\le N} \|u(n)\|_2$. Since X_N^m is finite-dimensional, it is a Banach space. Fix the history ϕ. Define $T_\phi : X_N^m \to X_N^m$ by $(T_\phi u)(0) := \phi(0)$ and, for $1 \le n \le N$, by

$$(T_\phi u)(n) := \phi(0) + \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{-\alpha} \times f\left(k, u(k), \widehat{u}_\phi(k-d)\right).$$

Because $0 < q\tau < 1$, we have $(r)_{q,\tau} = \frac{1-(q\tau)^r}{1-q} > 0$ for every $r = 1, \dots, N$. Hence, all kernel terms are positive and finite, and T_ϕ is well defined. Let $u, v \in X_N^m$. Since they are compared using the same prescribed history ϕ, whenever $k - d \le 0$,

$$\widehat{u}_\phi(k-d) = \widehat{v}_\phi(k-d) = \phi(k-d).$$

If $k - d \ge 1$, then $\widehat{u}_\phi(k-d) - \widehat{v}_\phi(k-d) = u(k-d) - v(k-d)$. Therefore, in both cases,

$$\|\widehat{u}_\phi(k-d) - \widehat{v}_\phi(k-d)\|_2 \le \|u - v\|_{\infty,N}.$$

Using the Lipschitz condition (5.17), for $1 \le n \le N$, we obtain

$$\begin{aligned}\|(T_\phi u)(n) - (T_\phi v)(n)\|_2 &\le \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{-\alpha} \times \Big[\mathcal{L}_1 \|u(k) - v(k)\|_2 \\ &\quad + \mathcal{L}_2 \left\|\widehat{u}_\phi(k-d) - \widehat{v}_\phi(k-d)\right\|_2\Big] \\ &\le (L_1 + L_2) C_N \|u - v\|_{\infty,N}.\end{aligned}$$

At $n = 0$, $\|(T_\phi u)(0) - (T_\phi v)(0)\|_2 = 0$. Taking the maximum over $0 \le n \le N$, we obtain $\|T_\phi u - T_\phi v\|_{\infty,N} \le (L_1 + L_2) C_N \|u - v\|_{\infty,N}$. Condition (5.21) shows that T_ϕ is a contraction. Banach's fixed-point theorem therefore gives a unique fixed point $u \in X_N^m$. This fixed point is the unique mild solution of (5.19). We next prove continuous dependence on the history. Let u and v correspond to histories ϕ and ψ, respectively. Set $D := \|u - v\|_{\infty,N}$ and $h := \|\phi - \psi\|_{\infty,-d,0}$. Define $Z := \max\{D, h\}$. For every relevant delayed index $r = k - d$, we have $\|\widehat{u}_\phi(r) - \widehat{v}_\psi(r)\|_2 \le Z$. Indeed, if $r \le 0$, the left-hand side is bounded by h, while if $r \ge 1$, it is bounded by D. For $1 \le n \le N$, subtracting the two Volterra equations gives

$$\begin{aligned}\|u(n) - v(n)\|_2 &\le \|\phi(0) - \psi(0)\|_2 + \frac{1}{\Gamma_{q,\tau}(1-\alpha)} \sum_{k=0}^{n-1} [(n-k)_{q,\tau}]^{-\alpha} \\ &\quad \times \Big[\mathcal{L}_1 \|u(k) - v(k)\|_2 + \mathcal{L}_2 \left\|\widehat{u}_\phi(k-d) - \widehat{v}_\psi(k-d)\right\|_2\Big] \\ &\le h + (L_1 + L_2) C_N Z.\end{aligned}$$

At $n = 0$, $\|u(0) - v(0)\|_2 = \|\phi(0) - \psi(0)\|_2 \le h$. Consequently, we get $D \le h + (L_1 + L_2) C_N Z$. Since $Z = \max\{D, h\}$, it follows that $Z \le h + (L_1 + L_2) C_N Z$. Using $(L_1 + L_2) C_N < 1$, we obtain $Z \le \frac{h}{1-(L_1+L_2)C_N}$. Because $D \le Z$, we conclude that

$$\|u - v\|_{\infty,N} \le \frac{\|\phi - \psi\|_{\infty,-d,0}}{1 - (\mathcal{L}_1 + \mathcal{L}_2) C_N}.$$

This proves continuous dependence on the initial history. Finally, suppose that $f(n,0,0)=0$ for every $n\in\mathbb{N}_{N-1}$, and let $\phi(r)=0$, $-d\le r\le 0$. Substitution of $u(n)=0$ into (5.19) gives

$$0+\frac{1}{\Gamma_{q,\tau}(1-\alpha)}\sum_{k=0}^{n-1}[(n-k)_{q,\tau}]^{-\alpha}f(k,0,0)=0.$$

Thus, the zero sequence is a mild solution. Its uniqueness follows from the contraction argument. □

Theorem 5.4 (Finite-horizon well-posedness and disagreement bound for delayed (q,τ)-Volterra networks)**.** *Let $M\ge 2$ be the number of nodes and $p\ge 1$ the dimension of each node state. Let $0<\alpha<1$, $0<q<1$, $\tau>0$, $0<q\tau<1$, and let $d\in\mathbb{N}$, $N\in\mathbb{N}$. Let $L_G=(\ell_{ij})\in\mathbb{R}^{M\times M}$ be the symmetric positive-semidefinite Laplacian of a connected undirected graph, with eigenvalues $0=\lambda_1<\lambda_2\le\cdots\le\lambda_M$. Let $H\in\mathbb{R}^{p\times p}$ be the inner-coupling matrix and let $\kappa\ge 0$ be the coupling strength. Suppose that $f:\mathbb{R}^p\longrightarrow\mathbb{R}^p$ is globally Lipschitz:*

$$\|f(x)-f(y)\|_2\le\mathcal{L}_f\|x-y\|_2,\qquad x,y\in\mathbb{R}^p,\tag{5.24}$$

for some $\mathcal{L}_f\ge 0$. For each node i, prescribe an initial history $\phi_i:\{-d,-d+1,\dots,0\}\longrightarrow\mathbb{R}^p$. Define the history extension $\widehat{u}_{i,\phi_i}(\varsigma):=\begin{cases}\phi_i(\varsigma), & -d\le\varsigma\le 0,\\ u_i(\varsigma), & 1\le\varsigma\le N.\end{cases}$ Consider the delayed network Volterra system

$$u_i(n)=\phi_i(0)+\sum_{k=0}^{n-1}w_{n-k}^{(\alpha,q,\tau)}\left[f(u_i(k))-\kappa\sum_{j=1}^{M}\ell_{ij}H\widehat{u}_{j,\phi_j}(k-d)\right],\qquad 1\le n\le N,\tag{5.25}$$

where

$$w_{\varsigma}^{(\alpha,q,\tau)}:=\frac{[(\varsigma)_{q,\tau}]^{-\alpha}}{\Gamma_{q,\tau}(1-\alpha)},\qquad(\varsigma)_{q,\tau}:=\frac{1-(q\tau)^{\varsigma}}{1-q},\ \Gamma_{q,\tau}(1-\alpha)>0.\tag{5.26}$$

Define the current-state kernel constant

$$C_N^{(\alpha,q,\tau)}:=\max_{1\le n\le N}\sum_{k=0}^{n-1}w_{n-k}^{(\alpha,q,\tau)},\tag{5.27}$$

the delayed-solution kernel constant

$$C_{N,d}^{(\alpha,q,\tau)}:=\max_{1\le n\le N}\sum_{k=d+1}^{n-1}w_{n-k}^{(\alpha,q,\tau)},\tag{5.28}$$

and the delayed-history kernel constant

$$B_{N,d}^{(\alpha,q,\tau)}:=\max_{1\le n\le N}\sum_{k=0}^{\min\{n-1,d\}}w_{n-k}^{(\alpha,q,\tau)}.\tag{5.29}$$

An empty sum is understood to be zero. If

$$\eta_{N,d}^{(\alpha,q,\tau)}:=\mathcal{L}_fC_N^{(\alpha,q,\tau)}+\kappa\lambda_M\|H\|_2C_{N,d}^{(\alpha,q,\tau)}<1,\tag{5.30}$$

then system (5.25) *has a unique mild solution on* $\{0,1,\dots,N\}$. *Moreover, let* U *and* V *be solutions corresponding to the stacked histories* Φ *and* Ψ, *respectively. Then*

$$\|U-V\|_{\infty,N} \le \frac{1+\kappa\lambda_M\|H\|_2 B_{N,d}^{(\alpha,q,\tau)}}{1-\eta_{N,d}^{(\alpha,q,\tau)}}\,\|\Phi-\Psi\|_{\infty,-d,0}. \tag{5.31}$$

Finally, define $P := I_M - \frac{1}{M}\mathbf{1}_M\mathbf{1}_M^T$ *and* $E_\perp(n) := (P\otimes I_p)U(n)$. *Let*

$$h_\perp := \max_{-d\le\varsigma\le 0}\|(P\otimes I_p)\Phi(\varsigma)\|_2.$$

Then

$$\|E_\perp\|_{\infty,N} \le \frac{1+\kappa\lambda_M\|H\|_2 B_{N,d}^{(\alpha,q,\tau)}}{1-\eta_{N,d}^{(\alpha,q,\tau)}}\,h_\perp. \tag{5.32}$$

In particular, if the complete initial history is synchronized, namely $\phi_1(\varsigma)=\phi_2(\varsigma)=\cdots=\phi_M(\varsigma)$ *for every* $\varsigma=-d,\dots,0$, *then* $u_1(n)=u_2(n)=\cdots=u_M(n)$ *for every* $n=0,\dots,N$. *Hence, the synchronization manifold is invariant on the finite horizon.*

Proof. Define the stacked state and stacked history by

$$U(n) := \mathrm{col}\big(u_1(n),\dots,u_M(n)\big)\in\mathbb{R}^{Mp}$$

and

$$\Phi(\varsigma) := \mathrm{col}\big(\phi_1(\varsigma),\dots,\phi_M(\varsigma)\big),\qquad -d\le\varsigma\le 0.$$

Define

$$\mathcal{F}(U) := \mathrm{col}\big(f(u_1),\dots,f(u_M)\big).$$

The history extension of the stacked state is

$$\widehat{U}_\Phi(\varsigma) := \begin{cases}\Phi(\varsigma), & -d\le\varsigma\le 0,\\ U(\varsigma), & 1\le\varsigma\le N.\end{cases}$$

System (5.25) can be written as

$$U(n) = \Phi(0) + \sum_{k=0}^{n-1} w_{n-k}^{(\alpha,q,\tau)}\left[\mathcal{F}(U(k)) - \kappa(L_G\otimes H)\widehat{U}_\Phi(k-d)\right]. \tag{5.33}$$

Because $0<q\tau<1$, we have $(\varsigma)_{q,\tau}>0$, $\varsigma=1,\dots,N$. Together with $\Gamma_{q,\tau}(1-\alpha)>0$, this shows that every kernel weight $w_\varsigma^{(\alpha,q,\tau)}$ is positive and finite. Let

$$X_N^{Mp} := \big\{U:\{0,\dots,N\}\to\mathbb{R}^{Mp}\big\}$$

be equipped with

$$\|U\|_{\infty,N} := \max_{0\le n\le N}\|U(n)\|_2.$$

This is a finite-dimensional Banach space. For the fixed history Φ, define $T_\Phi : X_N^{Mp}\to X_N^{Mp}$ by

$$(T_\Phi U)(0) := \Phi(0)$$

and, for $1 \le n \le N$, by

$$(T_\Phi U)(n) := \Phi(0) + \sum_{k=0}^{n-1} w_{n-k}^{(\alpha,q,\tau)} \left[\mathcal{F}(U(k)) - \kappa(L_G \otimes H)\widehat{U}_\Phi(k-d) \right].$$

Let $U, V \in X_N^{Mp}$. The nodewise Lipschitz condition implies

$$\|\mathcal{F}(U) - \mathcal{F}(V)\|_2 \le L_f \|U - V\|_2. \tag{5.34}$$

Since L_G is symmetric, $\|L_G\|_2 = \lambda_M$, and therefore

$$\|L_G \otimes H\|_2 = \lambda_M \|H\|_2. \tag{5.35}$$

When $0 \le k \le d$, the delayed index satisfies $k - d \le 0$. Since the history is fixed, $\widehat{U}_\Phi(k-d) = \widehat{V}_\Phi(k-d)$, so these delayed terms do not contribute to the contraction difference. When $k \ge d+1$, $\widehat{U}_\Phi(k-d) - \widehat{V}_\Phi(k-d) = U(k-d) - V(k-d)$. Consequently, for $1 \le n \le N$,

$$\begin{aligned}
\|(T_\Phi U)(n) - (T_\Phi V)(n)\|_2 &\le \mathcal{L}_f \sum_{k=0}^{n-1} w_{n-k}^{(\alpha,q,\tau)} \|U(k) - V(k)\|_2 \\
&\quad + \kappa\lambda_M \|H\|_2 \sum_{k=d+1}^{n-1} w_{n-k}^{(\alpha,q,\tau)} \|U(k-d) - V(k-d)\|_2 \\
&\le \eta_{N,d}^{(\alpha,q,\tau)} \|U - V\|_{\infty,N}.
\end{aligned}$$

Taking the maximum over n gives

$$\|T_\Phi U - T_\Phi V\|_{\infty,N} \le \eta_{N,d}^{(\alpha,q,\tau)} \|U - V\|_{\infty,N}.$$

Since $\eta_{N,d}^{(\alpha,q,\tau)} < 1$, the operator T_Φ is a contraction. Banach's fixed-point theorem therefore gives a unique mild solution. We next establish continuous dependence on the history. Let U, V correspond to histories Φ, Ψ, respectively. Define $D := \|U - V\|_{\infty,N}$ and $h := \|\Phi - \Psi\|_{\infty,-d,0}$. For $0 \le k \le d$, $\left\|\widehat{U}_\Phi(k-d) - \widehat{V}_\Psi(k-d)\right\|_2 \le h$. For $k \ge d+1$, $\left\|\widehat{U}_\Phi(k-d) - \widehat{V}_\Psi(k-d)\right\|_2 \le D$. Subtracting the two stacked Volterra equations and applying the preceding estimates gives

$$\|U(n) - V(n)\|_2 \le h + L_f C_N^{(\alpha,q,\tau)} D + \kappa\lambda_M \|H\|_2 \left[B_{N,d}^{(\alpha,q,\tau)} h + C_{N,d}^{(\alpha,q,\tau)} D \right].$$

Taking the maximum over $n = 0, \dots, N$ yields

$$D \le \left[1 + \kappa\lambda_M \|H\|_2 B_{N,d}^{(\alpha,q,\tau)} \right] h + \eta_{N,d}^{(\alpha,q,\tau)} D.$$

Since $\eta_{N,d}^{(\alpha,q,\tau)} < 1$, rearrangement gives

$$D \le \frac{1 + \kappa\lambda_M \|H\|_2 B_{N,d}^{(\alpha,q,\tau)}}{1 - \eta_{N,d}^{(\alpha,q,\tau)}} h.$$

This proves (5.31). We finally establish the transverse-disagreement estimate. Let $J := \frac{1}{M}\mathbf{1}_M \mathbf{1}_M^T$, $P = I_M - J$. Since $L_G \mathbf{1}_M = 0$ and L_G is symmetric, $PL_G = L_G P = L_G$. Therefore, $(P \otimes I_p)(L_G \otimes$

$H) = L_G \otimes H$. Also, $(L_G \otimes H)U = (L_G \otimes H)(P \otimes I_p)U$. Because the same function f acts at every node, $\mathcal{F}\big((J \otimes I_p)U\big)$ belongs to the synchronization manifold. Hence,

$$(P \otimes I_p)\mathcal{F}\big((J \otimes I_p)U\big) = 0.$$

Using the Lipschitz property,

$$\begin{aligned}\|(P \otimes I_p)\mathcal{F}(U)\|_2 &\le \big\|\mathcal{F}(U) - \mathcal{F}\big((J \otimes I_p)U\big)\big\|_2 \\ &\le L_f\|(P \otimes I_p)U\|_2.\end{aligned}$$

Applying $P \otimes I_p$ to (5.33), separating the prescribed-history and delayed-solution contributions, and taking norms gives

$$\begin{aligned}\|E_\perp\|_{\infty,N} \le h_\perp &+ L_f C_N^{(\alpha,q,\tau)}\|E_\perp\|_{\infty,N} \\ &+ \kappa\lambda_M\|H\|_2\left[B_{N,d}^{(\alpha,q,\tau)}h_\perp + C_{N,d}^{(\alpha,q,\tau)}\|E_\perp\|_{\infty,N}\right].\end{aligned}$$

Therefore,

$$\big(1 - \eta_{N,d}^{(\alpha,q,\tau)}\big)\|E_\perp\|_{\infty,N} \le \left[1 + \kappa\lambda_M\|H\|_2 B_{N,d}^{(\alpha,q,\tau)}\right]h_\perp.$$

Dividing by $1 - \eta_{N,d}^{(\alpha,q,\tau)} > 0$ gives (5.32). If the complete history is synchronized, then $h_\perp = 0$. The disagreement estimate implies $\|E_\perp\|_{\infty,N} = 0$. Hence, $u_1(n) = \cdots = u_M(n)$ for every $n = 0, \ldots, N$. This proves invariance of the synchronization manifold. □

Example 5.1 (Discrete-time delayed (q,τ)-Volterra network)**.** Consider a network of three scalar oscillators with one-step delay:

$$\begin{aligned}U(n) = U(0) + \sum_{k=0}^{n-1} w_{n-k}^{(\alpha,q,\tau)}\Big[&-\sin\big(\widehat{U}(k-1)\big) \\ &- \kappa L_G\widehat{U}(k-1) - D_B\widehat{U}(k-1)\Big], \qquad 1 \le n \le N,\end{aligned} \tag{5.36}$$

where the sine is evaluated componentwise,

$$w_\varsigma^{(\alpha,q,\tau)} = \frac{[(\varsigma)_{q,\tau}]^{-\alpha}}{\Gamma_{q,\tau}(1-\alpha)}, \qquad (\varsigma)_{q,\tau} = \frac{1-(q\tau)^\varsigma}{1-q}.$$

The model parameters are $\alpha = 0.8$, $q = 0.5$, $\tau = 1.2$, $q\tau = 0.6 < 1$, $\kappa = 0.2$, $B = 1$, $D_B = \mathrm{diag}(1,0,0)$, $N = 50$. The graph Laplacian is

$$L_G = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The prescribed one-step history is $\widehat{U}(-1) = U(0) = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, and $\widehat{U}(r) = U(r)$, $r \ge 0$. Numerical procedure and convergence criterion The only infinite quantity evaluated in the finite-horizon

recursion is the product defining $\Gamma_{q,\tau}(1-\alpha)$. Let $\Gamma_{q,\tau}^{[K]}(1-\alpha)$ denote its approximation obtained by retaining the first K factors, and let $U^{[K]}$ denote the corresponding numerical solution. The a posteriori convergence indicator is defined by

$$\widehat{\mathcal{E}}_K := \max_{0\le n\le N} \|U^{[2K]}(n) - U^{[K]}(n)\|_\infty. \tag{5.37}$$

The computation is regarded as converged when $\widehat{\mathcal{E}}_K < 10^{-6}$. For the reported results, $K = 40$ is used. Truncation-error estimate using K_{ref}=200 as the reference calculation, define

$$\mathcal{E}_K := \max_{0\le n\le 50} \|U^{[K]}(n) - U^{[200]}(n)\|_\infty. \tag{5.38}$$

Thus, K=20 already satisfies the selected tolerance, while K=40 gives approximately double-

Table 6. Convergence and product-truncation error for the delayed network example.

K	$\Gamma_{q,\tau}^{[K]}(1-\alpha)$	$\mathcal{E}_K$
5	3.124166057433	2.7289×10^{-2}
10	3.159628182932	4.2106×10^{-4}
20	3.160190210208	1.0278×10^{-7}
40	3.160190347453	6.2172×10^{-15}
80	3.160190347453	$< 10^{-15}$

precision accuracy. For recursion residual, we have The numerical residual is defined by

$$\mathcal{R}_K := \max_{1\le n\le N} \left\| U^{[K]}(n) - U(0) - \sum_{k=0}^{n-1} w_{n-k}^{[K]} G^{[K]}(k-1) \right\|_\infty, \tag{5.39}$$

where

$$G^{[K]}(k-1) = -\sin\big(\widehat{U}^{[K]}(k-1)\big) - \kappa L_G \widehat{U}^{[K]}(k-1) - D_B \widehat{U}^{[K]}(k-1).$$

Because the finite-memory recursion is evaluated explicitly, the calculated residual is at the level of floating-point round-off. Recalculated states are follows: Table 6 confirms rapid convergence with respect to the product-truncation level. The maximum error is approximately 1.03×10^{-7} at $K = 20$ and reaches the level of double-precision rounding at $K = 40$. Table 7 shows an oscillatory transient caused by the one-step delay, followed by a substantial reduction in the state magnitudes over the finite simulation interval. The maximum state magnitude decreases from 2 initially to approximately 4.13×10^{-4} at $n = 50$. These results provide finite-horizon numerical evidence and are not interpreted as a proof of infinite-horizon asymptotic stability.

Example 5.2 (Delayed (q,τ)-Volterra synchronization on a complete graph)**.** Consider a delayed network of three scalar oscillators:

$$U(n) = U(0) + \sum_{k=0}^{n-1} w_{n-k}^{(\alpha,q,\tau)} \left[-\sin\big(\widehat{U}(k-1)\big) - \kappa L_G \widehat{U}(k-1) \right], \tag{5.40}$$

where

$$w_r^{(\alpha,q,\tau)} = \frac{[(r)_{q,\tau}]^{-\alpha}}{\Gamma_{q,\tau}(1-\alpha)}, \qquad (r)_{q,\tau} = \frac{1-(q\tau)^r}{1-q}.$$

Table 7. Computed states for the one-step delayed network.

n	$u_1(n)$	$u_2(n)$	$u_3(n)$	$\max_i \lvert u_i(n)\rvert$
0	2.000000	-2.000000	1.000000	2.000000
1	0.596842	-1.126436	0.454719	1.126436
2	-0.366566	-0.526647	0.080328	0.526647
5	0.642382	-0.435931	0.176181	0.642382
10	0.251411	-0.176883	0.056791	0.251411
20	0.033419	-0.028993	0.008200	0.033419
30	0.001724	-0.004175	0.000937	0.004175
40	-0.000867	-0.000470	0.000040	0.000867
50	-0.000413	-0.000025	-0.000020	0.000413

The parameters are $d = 1, \alpha = 0.8, q = 0.5, \tau = 1.2, q\tau = 0.6 < 1, \kappa = 0.2, N = 50$. The graph Laplacian is

$$L_G = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

The prescribed one-step history is $\widehat{U}(-1) = U(0) = \begin{pmatrix} 0.5 \\ 1.0 \\ -1.5 \end{pmatrix}$, and $\widehat{U}(\varsigma) = U(\varsigma)$, $\varsigma \geq 0$. The target synchronized state is $s(n) \equiv 0$. Let $\Gamma_{q,\tau}^{[K]}(1-\alpha)$ be the approximation obtained by retaining the first K factors of the infinite product defining the deformed Gamma function. Denote the corresponding numerical solution by $U^{[K]}$. We define the successive-truncation indicator

$$\widehat{\mathcal{E}}_K := \max_{0 \leq n \leq N} \|U^{[2K]}(n) - U^{[K]}(n)\|_\infty. \tag{5.41}$$

The computation is declared converged when $\widehat{\mathcal{E}}_K < 10^{-6}$. Truncation-error analysis

Using K_{ref}=200 as the reference level, define

$$\mathcal{E}_K := \max_{0 \leq n \leq 50} \|U^{[K]}(n) - U^{[200]}(n)\|_\infty. \tag{5.42}$$

The calculated errors are in Table 8. Thus, K=20 meets the prescribed tolerance, while K=40 gives approximately machine-precision accuracy. Numerical residual can be found as follows: Define

$$\mathcal{R}_K := \max_{1 \leq n \leq N} \left\| U^{[K]}(n) - U(0) - \sum_{k=0}^{n-1} w_{n-k}^{[K]} G^{[K]}(k-1) \right\|_\infty, \tag{5.43}$$

where

$$G^{[K]}(k-1) = -\sin\big(\widehat{U}^{[K]}(k-1)\big) - \kappa L_G \widehat{U}^{[K]}(k-1).$$

Since the finite-horizon memory sum is evaluated explicitly, the computed residual is at the level of floating-point roundoff. Thus, we get Here,

Table 8. Convergence and product-truncation errors for the delayed complete-graph network.

K	$\Gamma_{q,\tau}^{[K]}(1-\alpha)$	$\mathcal{E}_K$
5	3.124166057433	1.3959×10^{-2}
10	3.159628182932	2.1539×10^{-4}
20	3.160190210208	5.2576×10^{-8}
40	3.160190347453	3.1086×10^{-15}
80	3.160190347453	$< 10^{-15}$

Table 9. Computed states for the one-step delayed complete-graph network.

n	$u_1(n)$	$u_2(n)$	$u_3(n)$	$\max_i \lvert u_i(n)\rvert$	$e_\perp(n)$
0	0.500000	1.000000	-1.500000	1.500000	1.500000
1	0.205158	0.454719	-0.782213	0.782213	0.741434
2	0.002719	0.080328	-0.289381	0.289381	0.220603
5	0.095667	0.193677	-0.301791	0.301791	0.297643
10	0.025523	0.055830	-0.096933	0.096933	0.091739
20	0.003225	0.007694	-0.014663	0.014663	0.013415
30	0.000429	0.001160	-0.002494	0.002494	0.002192
40	0.000045	0.000163	-0.000426	0.000426	0.000353
50	0.000001	0.000020	-0.000074	0.000074	0.000057

$$e_\perp(n) = \max_{1\le i\le 3} |u_i(n) - \overline{u}(n)|, \qquad \overline{u}(n) = \frac{u_1(n) + u_2(n) + u_3(n)}{3}.$$

Table 8 demonstrates rapid convergence with respect to the product-truncation level. The error is approximately 5.26×10^{-8} at $K = 20$ and reaches double-precision rounding at $K = 40$. Table 9 shows an initial oscillatory response induced by the one-step delay, followed by a substantial reduction in both the state magnitude and transverse disagreement. At the final simulated time, the maximum state magnitude is approximately 7.42×10^{-5}, while the disagreement is approximately 5.66×10^{-5}. These results constitute finite-horizon numerical evidence and do not independently prove infinite-horizon asymptotic synchronization.

6. Conclusion

This study developed a finite-horizon framework for delayed discrete-time oscillator networks based on the (q, τ)-fractional nabla operator. The main contribution is a mathematically consistent Volterra formulation that explicitly incorporates fractional memory, deformation parameters, delay, network coupling, and the prescribed initial history. The theoretical significance of these results lies in converting the combined effects of fractional memory, deformation, delay, and network coupling into explicit and verifiable finite-horizon bounds. These bounds identify parameter regimes in which the network Volterra problem is mathematically well posed, its solution depends continuously on the prescribed history, and synchronization disagreement remains controlled. They also clarify the distinct roles of the fractional order, deformation

parameters, delay length, graph spectrum, coupling strength, and observation horizon. Consequently, the conditions provide an analytical basis for selecting admissible parameters and for distinguishing rigorously supported behavior from numerical behavior observed outside the guaranteed region. Sufficient conditions were established for existence, uniqueness, continuous dependence, synchronization-manifold invariance, and finite-horizon control of the transverse disagreement. Unlike the original global estimates, the revised conditions use finite kernel constants and therefore avoid divergent infinite-horizon kernel sums. The delayed-network estimates show explicitly how the fractional order, deformation parameters, delay length, graph structure, coupling strength, and simulation horizon enter the analytical bounds. The adaptive-pinning analysis was reformulated using the smallest eigenvalue of the complete controlled network matrix. This provides an appropriate network-level spectral condition and establishes invariance after finite attainment without requiring the controller gains to become unbounded. The numerical study included delayed networks, time-varying coupling, and pinning control. Convergence criteria, product-truncation error estimates, recursion-residual checks, and comparisons among supported, near-threshold, and unsupported parameter regimes were provided. The results demonstrate stable finite-horizon numerical behavior and decreasing disagreement for the tested configurations, while also showing that the sufficient analytical conditions can be conservative. The present conclusions are deliberately restricted to finite-horizon well-posedness, synchronization invariance, and disagreement control. Establishing infinite-horizon asymptotic synchronization would require an appropriate framework, a fractional comparison principle, and a resolvent analysis for the adopted (q, τ)-kernel.

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