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NEW FRACTIONAL HERMITE–HADAMARD AND FEJÉR INEQUALITIES INVOLVING THE GAUSSIAN HYPERGEOMETRIC FUNCTION VIA SAIGO OPERATORS WITH APPLICATIONS TO TRAPEZOIDAL RULE*

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Abstract In this paper, our aim is to explore some new classes of integral inequalities for convex functions, which include the bounds of Gaussian hypergeometric functions. Some Hermite–Hadamard and Fejér type inequalities via interval valued mappings and fractional Saigo integral operator are established. These findings are novel and a generalization of many known results in literature, which can be seen through the remarks presented in a table form. Some results can be retrieved under various settings of fractional operators. Moreover, to demonstrate a practical application of the results, we show how they can be effectively employed to estimate the errors in the numerical trapezoidal quadrature rule.

Keywords Gamma and hypergeometric functions, interval-valued Saigo integrals, Hermite–Hadamard and Fejér inequalities, numerical quadrature, error estimates, fractional derivatives.

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1. Introduction

The study of convexity and integral inequalities occupies a fundamental position in mathematical analysis and its wide range of applications. Among the most classical and influential results in

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this area is the Hermite–Hadamard inequality, which provides bounds for the integral mean of a convex function in terms of its endpoint values. This inequality originates from the pioneering work of Jacques Hadamard and has been extensively discussed in the literature [17, 29]. Its importance has motivated a vast amount of research aimed at developing new generalizations and extensions.

In recent years, considerable attention has been given to extending the Hermite–Hadamard inequality in different directions. One of the most notable developments is the incorporation of fractional calculus, which enables the formulation of inequalities in nonlocal and memory-dependent settings. For example, Sarikaya et al. [40] established Hermite–Hadamard inequalities using Riemann–Liouville fractional integrals. Similarly, Hai and Wang [22] obtained related results via Erdélyi–Kober fractional integrals. Ahmad et al. [8] further contributed by deriving various inequalities, including Hermite–Hadamard–Fejér and Pachpatte-type inequalities, through newly introduced fractional operators. In the set-valued setting, Almalki and Afzal [11] obtained new Hermite–Hadamard estimates for harmonical $\mathfrak{cr}$-h-convex functions by means of a generalized fractional integral operator.

Alongside these advancements, generalized fractional operators have gained increasing importance, since they unify several well-known fractional operators through appropriate parameter choices. For instance, Afzal et al. [4] employed novel fractional integral operators under totally interval-order relations to derive Hermite–Hadamard, Pachpatte and Fejér type inequalities, together with Hyers–Ulam stability results for $2D$-convex mappings. Among such generalized operators, the Saigo fractional integral operator, whose kernel involves the Gauss hypergeometric function, contains the Riemann–Liouville and Erdélyi–Kober operators as particular cases. This unifying nature makes it especially suitable for deriving more general and comprehensive results in convex analysis.

At the same time, interval-valued analysis has emerged as an effective framework for handling uncertainty, imprecision, and incomplete information. Interval-valued functions naturally arise in applications such as optimization and variational problems [35, 46], control theory, and numerical analysis. For example, in the numerical investigation of heat and mass transfer in three-dimensional MHD nanoliquid flows with inclined magnetization [20], the physical parameters are typically known only within tolerance ranges, so that the computed quantities are naturally interval-valued. In this direction, Zhou et al. [49] investigated fractional integral inclusions for interval-valued convex functions with exponential kernels, meanwhile, Sahoo et al. [37] derived Hermite–Hadamard inequalities for (h, m)-convex functions. Khan et al. [27] further obtained new variants of Hermite–Hadamard and Fejér type inequalities for the Godunova–Levin preinvex class of interval-valued functions.

A significant milestone in this field is the extension of classical inequalities to interval-valued functions. Budak et al. [14] developed fractional Hermite–Hadamard inequalities in this setting, laying the groundwork for further studies. Subsequent contributions include results by Shi et al. [42], Nwaeze et al. [32], and Zhang et al. [47], who explored various aspects of interval-valued inequalities and integral inclusions. Kulisch–Miranker type inclusions for a generalized class of Godunova–Levin stochastic processes were subsequently developed by Afzal et al. [3]. Moreover, Ali et al. [10] and Kalsoom et al. [24] extended these ideas to post-quantum calculus.

Further progress has been made in the study of generalized convexity for interval-valued functions. For instance, Zhao et al. [48] examined Hermite–Hadamard inequalities for coordinated h-convex functions, while Tariboon et al. [45] introduced related inclusion results via post-quantum calculus. Extensions to preinvex and $\hbar$-preinvex interval-valued functions were later investigated by Srivastava et al. [43], Lai et al. [28], and Tan et al. [44], highlighting the

flexibility of convexity concepts in this framework. Within the center–radius framework, Sahoo et al. [39] established Hermite–Hadamard, Fejér and Pachpatte type inequalities for $\mathfrak{cr}$-ordered preinvex functions, whereas Afzal et al. [7] obtained new estimates of well-known inequalities for (h_1, h_2)-Godunova–Levin functions by means of the center–radius order relation.

Interval-valued inequalities have also been investigated under fuzzy and stochastic settings. For instance, Liu et al. [30] derived Jensen and Hermite–Hadamard-type inequalities for log-s-convex and Cheng et al. [15] considered fractional quantum versions, while Afzal et al. [6] analyzed inequalities for stochastic processes. More recent contributions by Rholam and Barmaki [34] and Fahad et al. [17] further expand these results to broader contexts. Adrees et al. [1] presented novel unified variants of Ostrowski, Jensen and Hermite–Hadamard inequalities for generalized (η, p, h)-convex stochastic processes, and interval set inclusion relations for (h_1, h_2)-convex stochastic processes were exploited by Afzal et al. [2].

Finally, several works have addressed fractional and reverse inequalities in the interval-valued setting. Akram et al. [9] studied Hermite–Hadamard inequalities via q-integrals, while Bin-Mohsin et al. [31] investigated reverse inequalities. Demir [16] provided refinements of Jensen-type inequalities, and Bohner et al. [13] introduced results for diamond-alpha integrals. Additional contributions by Khan et al. [26], Afzal et al. [5], Kalsoom [25], Firdous et al. [18], who obtained structural refinements of discrete weighted Jensen and Hermite–Hadamard inequalities for (α, m)-convex mappings, and Shah et al. [41] further demonstrate the ongoing development and richness of this research area.

Novelty and significance

Despite various developments in the field of fractional calculus, most results to date are mainly achieved via classical or Riemann–Liouville type fractional operators. Combining the notion of interval-valued functions with the Saigo fractional operator and using it to obtain sharp Hermite–Hadamard type inequalities involving special functions such as Gamma, Beta, and Gauss hypergeometric functions, along with applications to quadrature formulas, is rather scarce in the literature.

The contributions and significance of this study are described as follows:

- The first significant contribution of our study is introducing a new interval-valued Saigo fractional integral operator which generalizes the theory of fractional integral inequalities in the context of interval-valued functions. Using this operator, one can derive various known inequalities as special cases by choosing suitable parameters (see [22, 40]).
- Unlike the vast majority of papers in the existing literature, which use inclusion order for comparison (see [14]), it is worth mentioning that this type of comparison relation is quite restrictive and may not allow comparison between some pairs of intervals, as shown in Example 3 of [19]. Consequently, alternative order structures have been proposed. Notably, Bhunia and Samanta [12] introduced a complete and consistent ordering for interval-valued numbers that resolves this issue. In the present study, we adopt a mathematically consistent order relation that guarantees meaningful comparisons.
- Employing the newly introduced interval-valued Saigo operator, we establish several novel forms of Hermite–Hadamard, midpoint, and weighted product inequalities for interval-valued convex mappings. These results extend known inequalities and provide sharper bounds due to the flexibility of the Saigo fractional framework.

- Another important feature is that the obtained bounds are expressed in terms of special functions such as Gamma, Beta, and Gauss hypergeometric functions. This formulation enhances the analytical structure of the inequalities and provides deeper insight into their behavior and sharpness in the fractional setting.
- As an application, we investigate numerical quadrature rules for interval-valued functions by deriving inclusion-type error estimates, particularly for the weighted trapezoidal rule. These results provide reliable bounds in the presence of uncertainty.
- Furthermore, the generality of the obtained results is demonstrated through remarks and examples. Many known results are recovered as special cases under suitable parameter choices, confirming the unifying nature of the proposed framework.

The layout of this article is as follows. In Section 1, we review the literature and present the motivation. Section 2 introduces the necessary definitions and concepts related to interval-valued functions and Saigo fractional operators. Section 3 contains the main results, including Hermite–Hadamard, Fejér, and product inequalities, along with their representations involving special functions and applications to numerical quadrature rules. Finally, Section 4 concludes the paper.

2. Preliminary concepts and definitions

In this section, we recall some necessary notions that are used in developing our main findings. These include interval arithmetic operations, classical and interval-valued convexity, Saigo fractional operators, and special functions such as the Gamma, Beta, and Gauss hypergeometric functions.

2.1. Special functions

We begin by recalling the definitions of fundamental special functions that play a central role in fractional calculus and arise naturally in the formulation of our results.

Definition 2.1 (Euler Gamma function)**.** The Euler Gamma function is defined for $\Re(z) > 0$ by the integral representation

$$\Gamma(z) = \int_0^\infty \eta^{z-1} e^{-\eta}\, d\eta.$$

It generalizes the factorial function in the sense that, for positive integers n, one has $\Gamma(n+1) = n!$. Moreover, for $\alpha > -1$, it satisfies the recurrence relation $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

Definition 2.2 (Beta function)**.** The Beta function, also known as the Euler integral of the first kind, is defined for $\Re(x) > 0$ and $\Re(y) > 0$ by

$$\mathcal{B}(x, y) = \int_0^1 \eta^{x-1}(1-\eta)^{y-1}\, d\eta.$$

It is closely related to the Gamma function through the identity

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Definition 2.3 (Gauss hypergeometric function)**.** The Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by the power series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1, $$

where $(a)_n$ denotes the Pochhammer symbol (rising factorial) given by

$$ (a)_0 = 1, \quad (a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad n \geq 1. $$

This function admits analytic continuation beyond the unit disk and plays a fundamental role in the kernel of Saigo-type fractional integral operators.

2.2. Interval arithmetic and the $\mathfrak{cr}$-order

We define the set $\mathcal{R}_{\mathcal{I}}$ to be the family of all nonempty closed intervals on the real line $\mathcal{R}$. An interval $\zeta_1 = [\underline{\zeta_1}, \overline{\zeta_1}]$ is said to be *positive* if $\underline{\zeta_1} > 0$. We denote by $\mathcal{R}_{\mathcal{I}}^{+}$ the collection of all positive intervals and by $\mathcal{R}^{+}$ the set of all positive real numbers.

$$ \zeta_1 + \zeta_2 = [\underline{\zeta_1}, \overline{\zeta_1}] + [\underline{\zeta_2}, \overline{\zeta_2}] = [\underline{\zeta_1} + \underline{\zeta_2}, \overline{\zeta_1} + \overline{\zeta_2}], $$

$$ \lambda\zeta_1 = \lambda[\underline{\zeta_1}, \overline{\zeta_1}] = \begin{cases} [\lambda\underline{\zeta_1}, \lambda\overline{\zeta_1}], & \lambda > 0, \\ \{0\}, & \lambda = 0, \\ [\lambda\overline{\zeta_1}, \lambda\underline{\zeta_1}], & \lambda < 0, \end{cases} $$

respectively. For nonnegative intervals (i.e., $\underline{\zeta_1}, \underline{\zeta_2} \geq 0$) the product is given by $\zeta_1\zeta_2 = [\underline{\zeta_1}\,\underline{\zeta_2},\ \overline{\zeta_1}\,\overline{\zeta_2}]$.

For an interval $\zeta_1 = [\underline{\zeta_1}, \overline{\zeta_1}] \in \mathcal{R}_{\mathcal{I}}$, the quantity $\zeta_{1_c} = \frac{\overline{\zeta_1} + \underline{\zeta_1}}{2}$ is called the *center* of ζ_1, and $\zeta_{1_r} = \frac{\overline{\zeta_1} - \underline{\zeta_1}}{2}$ is called the *radius* of ζ_1. The interval ζ_1 can be represented in center-radius form as

$$ \zeta_1 = \left\langle \frac{\overline{\zeta_1} + \underline{\zeta_1}}{2}, \frac{\overline{\zeta_1} - \underline{\zeta_1}}{2} \right\rangle = \langle \zeta_{1_c}, \zeta_{1_r} \rangle. $$

Definition 2.4 ([38])**.** Let $\zeta_1 = [\underline{\zeta_1}, \overline{\zeta_1}] = \langle \zeta_{1_c}, \zeta_{1_r} \rangle$ and $\zeta_2 = [\underline{\zeta_2}, \overline{\zeta_2}] = \langle \zeta_{2_c}, \zeta_{2_r} \rangle$ be intervals in $\mathcal{R}_{\mathcal{I}}$. The center–radius order (abbreviated as $\mathfrak{cr}$-order) is defined as follows:

$$ \zeta_1 \preceq_{\mathfrak{cr}} \zeta_2 \Leftrightarrow \begin{cases} \zeta_{1_c} < \zeta_{2_c}, & \text{if } \zeta_{1_c} \neq \zeta_{2_c}, \\ \zeta_{1_r} \leq \zeta_{2_r}, & \text{if } \zeta_{1_c} = \zeta_{2_c}. \end{cases} $$

It is obvious that, for any two intervals $\zeta_1, \zeta_2 \in \mathcal{R}_{\mathcal{I}}$, one of the inequalities $\zeta_1 \preceq_{\mathfrak{cr}} \zeta_2$ or $\zeta_2 \preceq_{\mathfrak{cr}} \zeta_1$ holds. The notion of Riemann integration for interval-valued functions was first introduced in [14]. The following results can be established:

Theorem 2.1 ([14])**.** *Let $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}$ be an interval-valued function such that $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$. The function $\mathcal{F}$ is said to be Riemann integrable on $[\zeta_1, \zeta_2]$ if and only if both $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are Riemann integrable on $[\zeta_1, \zeta_2]$, and*

$$ \int_{\zeta_1}^{\zeta_2} \mathcal{F}(\mathrm{x})\,\mathrm{dx} = \left[\int_{\zeta_1}^{\zeta_2} \underline{\mathcal{F}}(\mathrm{x})\,\mathrm{dx}, \int_{\zeta_1}^{\zeta_2} \overline{\mathcal{F}}(\mathrm{x})\,\mathrm{dx} \right]. $$

The collection of all interval-valued functions that are Riemann integrable on $[\zeta_1, \zeta_2]$ is denoted by $\mathcal{IR}_{([\zeta_1,\zeta_2])}$.

Theorem 2.2 ([38]). *Let $\mathcal{F}, \mathcal{G} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}^{+}$ be given by $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$ and $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$. If $\mathcal{F}, \mathcal{G} \in \mathcal{IR}_{([\zeta_1,\zeta_2])}$ and $\mathcal{F}(\mathrm{x}) \preceq_{\mathfrak{cr}} \mathcal{G}(\mathrm{x})$ for all $\mathrm{x} \in [\zeta_1, \zeta_2]$, then*

$$\int_{\zeta_1}^{\zeta_2} \mathcal{F}(\mathrm{x})\, \mathrm{dx} \preceq_{\mathfrak{cr}} \int_{\zeta_1}^{\zeta_2} \mathcal{G}(\mathrm{x})\, \mathrm{dx}.$$

For a deeper treatment of interval analysis, see [29] and [14].

Besides the $\mathfrak{cr}$-order we shall use the endpoint-wise order: For $\zeta_1, \zeta_2 \in \mathcal{R}_{\mathcal{I}}$ we write

$$\zeta_1 \leq_{\mathrm{LU}} \zeta_2 \quad \Longleftrightarrow \quad \underline{\zeta_1} \leq \underline{\zeta_2} \ \text{ and } \ \overline{\zeta_1} \leq \overline{\zeta_2}.$$

The following elementary facts are used repeatedly, in particular to justify the multiplication of two inequalities in the product-type results.

Lemma 2.1. *Let $A, B, C, D \in \mathcal{R}_{\mathcal{I}}$.*

(i) If $A \leq_{\mathrm{LU}} B$, then $A \preceq_{\mathfrak{cr}} B$.

(ii) If A, B, C, D are nonnegative intervals with $A \leq_{\mathrm{LU}} B$ and $C \leq_{\mathrm{LU}} D$, then $AC \leq_{\mathrm{LU}} BD$ and consequently $AC \preceq_{\mathfrak{cr}} BD$.

(iii) Both $\leq_{\mathrm{LU}}$ and $\preceq_{\mathfrak{cr}}$ are preserved under addition of intervals, under multiplication by a nonnegative scalar, and under subtraction of a fixed interval T (i.e., $A \preceq_{\mathfrak{cr}} B$ implies $A - T \preceq_{\mathfrak{cr}} B - T$).

Proof. (i) If $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$, then $A_{\mathfrak{c}} \leq B_{\mathfrak{c}}$. If $A_{\mathfrak{c}} < B_{\mathfrak{c}}$, then $A \preceq_{\mathfrak{cr}} B$. If $A_{\mathfrak{c}} = B_{\mathfrak{c}}$, then both endpoint inequalities must be equalities, so $A = B$ and $A_{\mathrm{r}} = B_{\mathrm{r}}$.

(ii) For nonnegative intervals the product is computed endpoint-wise, $AC = [\underline{A}\,\underline{C}, \overline{A}\,\overline{C}]$. Since $0 \leq \underline{A} \leq \underline{B}$ and $0 \leq \underline{C} \leq \underline{D}$, we get $\underline{A}\,\underline{C} \leq \underline{B}\,\underline{D}$; similarly $\overline{A}\,\overline{C} \leq \overline{B}\,\overline{D}$. The last assertion follows from (i).

(iii) For $\leq_{\mathrm{LU}}$ this is immediate. For $\preceq_{\mathfrak{cr}}$, note that $(A+C)_{\mathfrak{c}} = A_{\mathfrak{c}} + C_{\mathfrak{c}}$ and $(A+C)_{\mathrm{r}} = A_{\mathrm{r}} + C_{\mathrm{r}}$, $(\lambda A)_{\mathfrak{c}} = \lambda A_{\mathfrak{c}}$, $(\lambda A)_{\mathrm{r}} = \lambda A_{\mathrm{r}}$ for $\lambda \geq 0$, and $(A - T)_{\mathfrak{c}} = A_{\mathfrak{c}} - T_{\mathfrak{c}}$, $(A - T)_{\mathrm{r}} = A_{\mathrm{r}} + T_{\mathrm{r}}$; a direct case distinction on the centers gives the claim. □

We emphasize that the $\mathfrak{cr}$-order by itself is *not* compatible with interval multiplication. For example, with $A = [0.1, 3.8]$, $B = [1.99, 2.01]$ and $C = [0.1, 3.9]$ we have $A \preceq_{\mathfrak{cr}} B$ (since $A_{\mathfrak{c}} = 1.95 < 2 = B_{\mathfrak{c}}$), but $AC = [0.01, 14.82]$ has center 7.415, whereas $BC = [0.199, 7.839]$ has center 4.019, so $AC \not\preceq_{\mathfrak{cr}} BC$. This is the reason why the product inequalities in Section 3 are derived via Lemma 2.1(ii) from the endpoint-wise form of convexity (Definition 2.6).

2.3. Convexity and interval-valued convexity

Definition 2.5 ([29]). A function $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}$ is said to be convex if for all $\mathrm{x}, \mathfrak{y} \in [\zeta_1, \zeta_2]$ and $\lambda \in [0, 1]$, we have

$$\mathcal{F}(\lambda \mathrm{x} + (1 - \lambda)\mathfrak{y}) \leq \lambda \mathcal{F}(\mathrm{x}) + (1 - \lambda)\mathcal{F}(\mathfrak{y}).$$

Definition 2.6 ([38]). Suppose $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}$ is an interval-valued function defined by $\mathcal{F} = [\underline{\mathcal{F}}, \overline{\mathcal{F}}]$. Then $\mathcal{F}$ is called interval-valued convex on $[\zeta_1, \zeta_2]$ if both $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are convex functions on $[\zeta_1, \zeta_2]$ (and, of course, $\underline{\mathcal{F}}(\mathrm{x}) \leq \overline{\mathcal{F}}(\mathrm{x})$ for all $\mathrm{x} \in [\zeta_1, \zeta_2]$); equivalently, for every $\mathrm{x}, \mathfrak{y} \in [\zeta_1, \zeta_2]$ and $\lambda \in [0, 1]$,

$$\mathcal{F}(\lambda \mathrm{x} + (1 - \lambda)\mathfrak{y}) \leq_{\mathrm{LU}} \lambda \mathcal{F}(\mathrm{x}) + (1 - \lambda)\mathcal{F}(\mathfrak{y}).$$

By Lemma 2.1(i), every interval-valued convex function satisfies

$$\mathcal{F}(\lambda \mathrm{x} + (1-\lambda)\mathfrak{y}) \preceq_{\mathfrak{cr}} \lambda \mathcal{F}(\mathrm{x}) + (1-\lambda)\mathcal{F}(\mathfrak{y}).$$

The set of all interval-valued convex functions on $[\zeta_1, \zeta_2]$ with values in $\mathcal{R}_{\mathcal{I}}^{+}$ is denoted by $\mathcal{IVCX}\big([\zeta_1, \zeta_2], \mathcal{R}_{\mathcal{I}}^{+}\big)$.

Definition 2.7 ([38]). Let $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}$ be an interval-valued function represented in center-radius form as $\mathcal{F} = \langle \mathcal{F}_{\mathfrak{c}}, \mathcal{F}_{\mathrm{r}} \rangle$, where $\mathcal{F}_{\mathfrak{c}}(\mathrm{x}) = \frac{\overline{\mathcal{F}}(\mathrm{x}) + \underline{\mathcal{F}}(\mathrm{x})}{2}$ and $\mathcal{F}_{\mathrm{r}}(\mathrm{x}) = \frac{\overline{\mathcal{F}}(\mathrm{x}) - \underline{\mathcal{F}}(\mathrm{x})}{2}$. Then $\mathcal{F}$ is said to be $\mathfrak{cr}$-convex on $[\zeta_1, \zeta_2]$ if for all $\mathrm{x}, \mathfrak{y} \in [\zeta_1, \zeta_2]$ and $\lambda \in [0,1]$, we have

$$\mathcal{F}_{\mathfrak{c}}(\lambda \mathrm{x} + (1-\lambda)\mathfrak{y}) \leq \lambda \mathcal{F}_{\mathfrak{c}}(\mathrm{x}) + (1-\lambda)\mathcal{F}_{\mathfrak{c}}(\mathfrak{y}),$$

and

$$\mathcal{F}_{\mathrm{r}}(\lambda \mathrm{x} + (1-\lambda)\mathfrak{y}) \leq \lambda \mathcal{F}_{\mathrm{r}}(\mathrm{x}) + (1-\lambda)\mathcal{F}_{\mathrm{r}}(\mathfrak{y}).$$

Remark 2.1. For an interval-valued function $\mathcal{F}$, the following relations hold (see also [38,39]):

- If $\mathcal{F}$ is interval-valued convex in the sense of Definition 2.6, then $\mathcal{F}$ satisfies the convexity inequality with respect to $\preceq_{\mathfrak{cr}}$. This is Lemma 2.1(i) applied to the endpoint-wise inequality.
- If $\mathcal{F}$ is $\mathfrak{cr}$-convex (Definition 2.7), then $\mathcal{F}$ also satisfies the convexity inequality with respect to $\preceq_{\mathfrak{cr}}$. Indeed, put $P = \mathcal{F}(\lambda \mathrm{x} + (1-\lambda)\mathfrak{y})$ and $Q = \lambda \mathcal{F}(\mathrm{x}) + (1-\lambda)\mathcal{F}(\mathfrak{y})$. Then $Q_{\mathfrak{c}} = \lambda \mathcal{F}_{\mathfrak{c}}(\mathrm{x}) + (1-\lambda)\mathcal{F}_{\mathfrak{c}}(\mathfrak{y}) \geq P_{\mathfrak{c}}$ by convexity of $\mathcal{F}_{\mathfrak{c}}$; if $P_{\mathfrak{c}} < Q_{\mathfrak{c}}$ we are done, and if $P_{\mathfrak{c}} = Q_{\mathfrak{c}}$, then $P_{\mathrm{r}} \leq Q_{\mathrm{r}}$ by convexity of $\mathcal{F}_{\mathrm{r}}$. Hence $P \preceq_{\mathfrak{cr}} Q$.
- $\mathfrak{cr}$-convexity does not imply convexity of the endpoint functions. For instance, on $[0,1]$ let $\mathcal{F}_{\mathfrak{c}}(\mathrm{x}) = 3 + \mathrm{x}^2$ and $\mathcal{F}_{\mathrm{r}}(\mathrm{x}) = 2\mathrm{x}^2$; both are convex, but $\mathcal{F}(\mathrm{x}) = [3 - \mathrm{x}^2,\ 3 + 3\mathrm{x}^2]$ has a concave lower endpoint. For this reason, the results of Section 3 are formulated for interval-valued convex functions in the sense of Definition 2.6.

2.4. Saigo fractional integral operators

Definition 2.8 (Saigo fractional integral [33]). Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F} \in \mathcal{R}_{[\zeta_1, \zeta_2]}$. For parameters $\alpha > 0$ and $\beta, \gamma \in \mathcal{R}$, the left- and right-sided Saigo fractional integral operators are defined by

$$\mathcal{I}_{\zeta_1^+}^{\alpha,\beta,\gamma}\mathcal{F}(\mathrm{x}) = \frac{\mathrm{x}^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{\zeta_1}^{\mathrm{x}} (\mathrm{x}-\tau)^{\alpha-1}\, {}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; 1 - \frac{\tau}{\mathrm{x}}\Big)\, \mathcal{F}(\tau)\, d\tau, \quad \mathrm{x} > \zeta_1,$$

and

$$\mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\mathcal{F}(\mathrm{x}) = \frac{1}{\Gamma(\alpha)} \int_{\mathrm{x}}^{\zeta_2} (\tau-\mathrm{x})^{\alpha-1}\, \tau^{-\alpha-\beta}\, {}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; 1 - \frac{\mathrm{x}}{\tau}\Big)\, \mathcal{F}(\tau)\, d\tau, \quad \mathrm{x} < \zeta_2,$$

where $\Gamma(\cdot)$ denotes the Euler Gamma function and ${}_2F_1(\cdot)$ is the Gauss hypergeometric function.

Remark 2.2. The Saigo fractional integral operator defined in Definition 2.8 reduces to several classical fractional operators as special cases.

- **Riemann–Liouville fractional integral.** If $\beta = -\alpha$ and $\gamma = 0$, then

$$\mathcal{I}_{\zeta_1^+}^{\alpha,-\alpha,0}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}\int_{\zeta_1}^{x}(x-\tau)^{\alpha-1}\mathcal{F}(\tau)\,d\tau,$$

  which coincides with the classical Riemann–Liouville fractional integral operator [14, 40].

- **Erdélyi–Kober fractional integral.** If $\beta = 0$, then

$$\mathcal{I}_{\zeta_1^+}^{\alpha,0,\gamma}\mathcal{F}(x) = \frac{x^{-\alpha-\gamma}}{\Gamma(\alpha)}\int_{\zeta_1}^{x}(x-\tau)^{\alpha-1}\tau^{\gamma}\mathcal{F}(\tau)\,d\tau,$$

  which corresponds to the Erdélyi–Kober fractional integral operator [22].

3. The major results

In this section, we present the main results of the paper. Before proceeding to these results, we first extend the classical Saigo fractional integral operator to the framework of interval-valued functions. Using this extension, we then establish several new variants of Hermite–Hadamard and midpoint type inequalities for interval-valued convex functions.

Definition 3.1 (Interval-valued Saigo fractional integrals)**.** Let $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}$ be an interval-valued function defined by

$$\mathcal{F}(x) = \big[\,\underline{\mathcal{F}}(x),\, \overline{\mathcal{F}}(x)\,\big],$$

where $\underline{\mathcal{F}}, \overline{\mathcal{F}} \in \mathcal{R}_{[\zeta_1,\zeta_2]}$.

For parameters $\alpha > 0$ and $\beta, \gamma \in \mathcal{R}$ satisfying condition (H) of Remark 3.1 below, the interval-valued Saigo fractional integrals are defined componentwise as:

$$\mathcal{I}_{\zeta_1^+}^{\alpha,\beta,\gamma}\mathcal{F}(x) = \Big[\mathcal{I}_{\zeta_1^+}^{\alpha,\beta,\gamma}\underline{\mathcal{F}}(x),\ \mathcal{I}_{\zeta_1^+}^{\alpha,\beta,\gamma}\overline{\mathcal{F}}(x)\Big],$$
$$\mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\mathcal{F}(x) = \Big[\mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\underline{\mathcal{F}}(x),\ \mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\overline{\mathcal{F}}(x)\Big].$$

Explicitly, these operators can be written as:

$$\mathcal{I}_{\zeta_1^+}^{\alpha,\beta,\gamma}\mathcal{F}(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_{\zeta_1}^{x}(x-\tau)^{\alpha-1}\,{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;1-\frac{\tau}{x}\Big)\,\mathcal{F}(\tau)\,d\tau,\quad x>\zeta_1,$$
$$\mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\mathcal{F}(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\zeta_2}(\tau-x)^{\alpha-1}\,\tau^{-\alpha-\beta}\,{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;1-\frac{x}{\tau}\Big)\,\mathcal{F}(\tau)\,d\tau,\quad x<\zeta_2,$$

where ${}_2F_1(\cdot)$ denotes the Gauss hypergeometric function and the integrals are understood in the interval sense.

Remark 3.1 (Well-posedness of Definition 3.1)**.** Since $0 < \zeta_1 \le \tau < x$ in the left-sided operator (respectively $0 < x < \tau \le \zeta_2$ in the right-sided one), the argument of the hypergeometric function lies in $[0, 1)$, so the series converges. Throughout Section 3 we assume

$$\text{(H)}\qquad {}_2F_1(\alpha+\beta,-\gamma;\alpha;z) \ge 0 \quad \text{for all } z \in [0,1).$$

Condition (H) holds, for example, when $\gamma \le 0$ and $\alpha + \beta \ge 0$ (all Taylor coefficients are then nonnegative), and when $\gamma = 1$ and $\beta \le 0$ (then ${}_2F_1(\alpha+\beta,-1;\alpha;z) = 1 - \frac{\alpha+\beta}{\alpha}z \ge 1 - z > 0$

if $\alpha+\beta\geq 0$, and ≥ 1 otherwise); in particular, it holds in all examples below. Under (H) the whole Saigo kernel is nonnegative, so $\underline{\mathcal{F}}\leq\overline{\mathcal{F}}$ and the monotonicity of the Lebesgue integral give $\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\underline{\mathcal{F}}(\mathrm{x})\leq\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\overline{\mathcal{F}}(\mathrm{x})$ and $\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\underline{\mathcal{F}}(\mathrm{x})\leq\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\overline{\mathcal{F}}(\mathrm{x})$; hence Definition 3.1 indeed produces intervals. Nonnegativity of the kernels is also what allows us to multiply order relations by the kernels and integrate them in the proofs below.

3.1. Generalized Hermite–Hadamard type inequalities via interval-valued Saigo fractional integral operators

Theorem 3.1. *Let $\zeta_1,\zeta_2\in\mathcal{R}$ with $0<\zeta_1<\zeta_2$, and let $\mathcal{F}:[\zeta_1,\zeta_2]\to\mathcal{R}^+_{\mathcal{I}}$ be an interval-valued convex function represented as*

$$\mathcal{F}(\zeta)=\big[\underline{\mathcal{F}}(\zeta),\overline{\mathcal{F}}(\zeta)\big],$$

where $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are Lebesgue integrable on $[\zeta_1,\zeta_2]$ and satisfy

$$0\leq\underline{\mathcal{F}}(\zeta)\leq\overline{\mathcal{F}}(\zeta),\qquad\zeta\in[\zeta_1,\zeta_2].$$

Let $\alpha>0$ and $\beta,\gamma\in\mathcal{R}$ satisfy (H)*, and write $\Delta:=\zeta_2-\zeta_1$. Then*

$$\begin{aligned}&\left(\int_0^1\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)\,d\lambda\right)\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right)\\&\preceq_{\mathrm{cr}}\frac{1}{2}\left[\frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^\alpha}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\mathcal{F}(\zeta_2)+\int_0^1\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)\mathcal{F}(\zeta_1+\lambda\Delta)\,d\lambda\right],\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&\left(\int_0^1\lambda^{\alpha-1}(\zeta_1+\lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right)\,d\lambda\right)\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right)\\&\preceq_{\mathrm{cr}}\frac{1}{2}\left[\int_0^1\lambda^{\alpha-1}(\zeta_1+\lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right)\mathcal{F}(\zeta_2-\lambda\Delta)\,d\lambda\right.\\&\left.+\frac{\Gamma(\alpha)}{\Delta^\alpha}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\mathcal{F}(\zeta_1)\right].\end{aligned}\tag{3.2}$$

Consequently,

$$\begin{aligned}&\left[\int_0^1\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)\,d\lambda\right.\\&\left.+\int_0^1\lambda^{\alpha-1}(\zeta_1+\lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right)\,d\lambda\right]\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right)\\&\preceq_{\mathrm{cr}}\frac{1}{2}\left[\frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^\alpha}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\mathcal{F}(\zeta_2)+\frac{\Gamma(\alpha)}{\Delta^\alpha}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\mathcal{F}(\zeta_1)\right.\\&+\int_0^1\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)\mathcal{F}(\zeta_1+\lambda\Delta)\,d\lambda\\&\left.+\int_0^1\lambda^{\alpha-1}(\zeta_1+\lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right)\mathcal{F}(\zeta_2-\lambda\Delta)\,d\lambda\right].\end{aligned}\tag{3.3}$$

Moreover, by convexity,

$$\begin{aligned}
&\int_0^1 \lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)\mathcal{F}(\zeta_1+\lambda\Delta)\,d\lambda\\
&\quad+\int_0^1 \lambda^{\alpha-1}(\zeta_1+\lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right)\mathcal{F}(\zeta_2-\lambda\Delta)\,d\lambda\\
\preceq_{\mathrm{cr}}&\left[\int_0^1 (1-\lambda)\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)d\lambda\right.\\
&\quad\left.+\int_0^1 \lambda^{\alpha}(\zeta_1+\lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right)d\lambda\right]\mathcal{F}(\zeta_1)\\
&\quad+\left[\int_0^1 \lambda^{\alpha}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)d\lambda\right.\\
&\quad\left.+\int_0^1 (1-\lambda)\lambda^{\alpha-1}(\zeta_1+\lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right)d\lambda\right]\mathcal{F}(\zeta_2).
\end{aligned}\tag{3.4}$$

Proof. By convexity of $\mathcal{F}$, for every $\lambda\in[0,1]$ we have

$$\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right)=\mathcal{F}\left(\frac{(\lambda\zeta_1+(1-\lambda)\zeta_2)+((1-\lambda)\zeta_1+\lambda\zeta_2)}{2}\right)$$

and hence

$$\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right)\preceq_{\mathrm{cr}}\frac{1}{2}\Big[\mathcal{F}(\lambda\zeta_1+(1-\lambda)\zeta_2)+\mathcal{F}((1-\lambda)\zeta_1+\lambda\zeta_2)\Big].$$

Multiplying both sides by the kernel

$$\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right),$$

which is nonnegative by (H), and integrating over $[0,1]$ (Lemma 2.1(iii) and Theorem 2.2), we get

$$\begin{aligned}
&\left(\int_0^1 \lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)d\lambda\right)\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right)\\
\preceq_{\mathrm{cr}}&\frac{1}{2}\left[\int_0^1 \lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)\mathcal{F}(\lambda\zeta_1+(1-\lambda)\zeta_2)\,d\lambda\right.\\
&\quad\left.+\int_0^1 \lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)\mathcal{F}((1-\lambda)\zeta_1+\lambda\zeta_2)\,d\lambda\right].
\end{aligned}$$

Recalling that

$$\lambda\zeta_1+(1-\lambda)\zeta_2=\zeta_2-\lambda\Delta,$$

we now use the substitution

$$u=\zeta_2-\lambda\Delta,\qquad \lambda=\frac{\zeta_2-u}{\Delta},\qquad d\lambda=-\frac{1}{\Delta}\,du.$$

When $\lambda = 0$, we have $u = \zeta_2$, and when $\lambda = 1$, we have $u = \zeta_1$. Therefore,

$$\begin{aligned}&\int_0^1 \lambda^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_2}\right) \mathcal{F}(\zeta_2 - \lambda\Delta)\, d\lambda \\ &= \frac{1}{\Delta^\alpha} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - u)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1 - \frac{u}{\zeta_2}\right) \mathcal{F}(u)\, du.\end{aligned}$$

By the definition of the left-sided Saigo fractional integral at ζ_2,

$$\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\mathcal{F}(\zeta_2) = \frac{\zeta_2^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - u)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1 - \frac{u}{\zeta_2}\right) \mathcal{F}(u)\, du.$$

Hence,

$$\begin{aligned}&\int_0^1 \lambda^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_2}\right) \mathcal{F}(\zeta_2 - \lambda\Delta)\, d\lambda \\ &= \frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^\alpha} \mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\mathcal{F}(\zeta_2).\end{aligned}$$

Also, since

$$(1-\lambda)\zeta_1 + \lambda\zeta_2 = \zeta_1 + \lambda\Delta,$$

the remaining term is

$$\int_0^1 \lambda^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_2}\right) \mathcal{F}(\zeta_1 + \lambda\Delta)\, d\lambda.$$

Substituting these two expressions into the previous inequality yields (3.1).

Similarly, multiplying the convexity inequality by the nonnegative kernel

$$\lambda^{\alpha-1}(\zeta_1 + \lambda\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1 + \lambda\Delta}\right)$$

and integrating over $[0,1]$, we obtain

$$\begin{aligned}&\left(\int_0^1 \lambda^{\alpha-1}(\zeta_1 + \lambda\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1 + \lambda\Delta}\right) d\lambda\right) \mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right) \\ \preceq_{\mathrm{cr}}\ &\frac{1}{2}\left[\int_0^1 \lambda^{\alpha-1}(\zeta_1 + \lambda\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1 + \lambda\Delta}\right) \mathcal{F}(\zeta_2 - \lambda\Delta)\, d\lambda\right. \\ &\left. + \int_0^1 \lambda^{\alpha-1}(\zeta_1 + \lambda\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1 + \lambda\Delta}\right) \mathcal{F}(\zeta_1 + \lambda\Delta)\, d\lambda\right].\end{aligned}$$

Now, using the substitution

$$v = \zeta_1 + \lambda\Delta, \qquad \lambda = \frac{v - \zeta_1}{\Delta}, \qquad d\lambda = \frac{1}{\Delta}\, dv,$$

with $v \in [\zeta_1, \zeta_2]$, the second integral becomes

$$\int_0^1 \lambda^{\alpha-1}(\zeta_1 + \lambda\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1 + \lambda\Delta}\right) \mathcal{F}(\zeta_1 + \lambda\Delta)\, d\lambda$$

$$= \frac{1}{\Delta^{\alpha}} \int_{\zeta_1}^{\zeta_2} (v-\zeta_1)^{\alpha-1} v^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{\zeta_1}{v}\right) \mathcal{F}(v)\, dv.$$

Recalling the definition of the right-sided Saigo fractional integral at ζ_1,

$$\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\mathcal{F}(\zeta_1) = \frac{1}{\Gamma(\alpha)} \int_{\zeta_1}^{\zeta_2} (v-\zeta_1)^{\alpha-1} v^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{\zeta_1}{v}\right) \mathcal{F}(v)\, dv,$$

we conclude that

$$\begin{aligned} &\int_0^1 \lambda^{\alpha-1} (\zeta_1+\lambda\Delta)^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right) \mathcal{F}(\zeta_1+\lambda\Delta)\, d\lambda \\ &= \frac{\Gamma(\alpha)}{\Delta^{\alpha}} \mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\mathcal{F}(\zeta_1). \end{aligned}$$

Substituting this into the preceding estimate yields (3.2).

Adding (3.1) and (3.2) (Lemma 2.1(iii)), we directly obtain (3.3).

To prove (3.4), we again use convexity. Since

$$\mathcal{F}(\zeta_1+\lambda\Delta) = \mathcal{F}((1-\lambda)\zeta_1+\lambda\zeta_2) \preceq_{\mathrm{cr}} (1-\lambda)\mathcal{F}(\zeta_1) + \lambda\mathcal{F}(\zeta_2),$$

and

$$\mathcal{F}(\zeta_2-\lambda\Delta) = \mathcal{F}(\lambda\zeta_1+(1-\lambda)\zeta_2) \preceq_{\mathrm{cr}} \lambda\mathcal{F}(\zeta_1) + (1-\lambda)\mathcal{F}(\zeta_2),$$

we multiply the first inequality by

$$\lambda^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_2}\right)$$

and the second by

$$\lambda^{\alpha-1} (\zeta_1+\lambda\Delta)^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right),$$

then integrate over $[0,1]$. This gives

$$\begin{aligned} &\int_0^1 \lambda^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_2}\right) \mathcal{F}(\zeta_1+\lambda\Delta)\, d\lambda \\ &+ \int_0^1 \lambda^{\alpha-1} (\zeta_1+\lambda\Delta)^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right) \mathcal{F}(\zeta_2-\lambda\Delta)\, d\lambda \\ \preceq_{\mathrm{cr}} &\left[\int_0^1 (1-\lambda)\lambda^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_2}\right) d\lambda \right. \\ &\left. + \int_0^1 \lambda^{\alpha} (\zeta_1+\lambda\Delta)^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right) d\lambda\right] \mathcal{F}(\zeta_1) \\ &+ \left[\int_0^1 \lambda^{\alpha} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_2}\right) d\lambda \right. \\ &\left. + \int_0^1 (1-\lambda)\lambda^{\alpha-1} (\zeta_1+\lambda\Delta)^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right) d\lambda\right] \mathcal{F}(\zeta_2), \end{aligned}$$

which is exactly (3.4). This completes the proof. □

Example 3.1. Under the assumptions of Theorem 3.1, consider

$$\mathcal{F} : [1,3] \to \mathcal{R}_{\mathcal{I}}^{+}, \qquad \mathcal{F}(y) = \big[\underline{\mathcal{F}}(y), \overline{\mathcal{F}}(y)\big] = \big[y^2+1,\, y^3+2\big].$$

Clearly,

$$0 \le \underline{\mathcal{F}}(y) = y^2+1 \le y^3+2 = \overline{\mathcal{F}}(y), \qquad y \in [1,3],$$

and both endpoint functions are Lebesgue integrable on $[1,3]$. Moreover, since

$$\underline{\mathcal{F}}''(y) = 2 > 0, \qquad \overline{\mathcal{F}}''(y) = 6y > 0, \qquad y \in [1,3],$$

both $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are convex on $[1,3]$. Hence, $\mathcal{F}$ is interval-valued convex on $[1,3]$.

Choose

$$\alpha = 2, \qquad \beta = 0, \qquad \gamma = 1, \qquad \zeta_1 = 1, \qquad \zeta_2 = 3.$$

Then

$$\Delta = \zeta_2 - \zeta_1 = 2.$$

Also, since

$${}_2F_1(2,-1;2;z) = 1 - z,$$

the hypergeometric kernels appearing in Theorem 3.1 become explicit and nontrivial; moreover, $1 - z > 0$ on $[0,1)$, so condition (H) holds.

The midpoint term is

$$\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right) = \mathcal{F}(2) = [5,10].$$

We first evaluate the coefficient multiplying the midpoint term in (3.1). Since

$$\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right) = \lambda\,{}_2F_1\left(2,-1;2;\frac{2\lambda}{3}\right) = \lambda\left(1-\frac{2\lambda}{3}\right),$$

we obtain

$$\begin{aligned}\int_0^1 \lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)\,d\lambda &= \int_0^1 \lambda\left(1-\frac{2\lambda}{3}\right)\,d\lambda \\ &= \int_0^1 \lambda\,d\lambda - \frac{2}{3}\int_0^1 \lambda^2\,d\lambda \\ &= \frac{1}{2} - \frac{2}{9} \\ &= \frac{5}{18}.\end{aligned}$$

Therefore,

$$\left(\int_0^1 \lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_2}\right)\,d\lambda\right)\mathcal{F}(2) = \frac{5}{18}[5,10] = \left[\frac{25}{18},\frac{25}{9}\right].$$

Next, consider the left-sided Saigo fractional integral at $\zeta_2 = 3$. By definition,

$$\mathcal{I}_{1^+}^{2,0,1}\mathcal{F}(3) = \frac{3^{-2}}{\Gamma(2)}\int_1^3 (3-u)\,{}_2F_1\left(2,-1;2;1-\frac{u}{3}\right)\mathcal{F}(u)\,du.$$

Since $\Gamma(2) = 1$ and

$$ {}_2F_1\Big(2, -1; 2; 1 - \frac{u}{3}\Big) = 1 - \Big(1 - \frac{u}{3}\Big) = \frac{u}{3}, $$

we get

$$ \mathcal{I}_{1^+}^{2,0,1}\mathcal{F}(3) = \frac{1}{9}\int_1^3 (3-u)\frac{u}{3}\mathcal{F}(u)\,du = \frac{1}{27}\int_1^3 u(3-u)\mathcal{F}(u)\,du. $$

Hence,

$$ \mathcal{I}_{1^+}^{2,0,1}\mathcal{F}(3) = \left[\frac{224}{405}, \frac{458}{405}\right]. $$

The corresponding coefficient in (3.1) is

$$ \frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^\alpha} = \frac{\Gamma(2)\,3^2}{2^2} = \frac{9}{4}. $$

Therefore,

$$ \frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^\alpha}\mathcal{I}_{1^+}^{2,0,1}\mathcal{F}(3) = \frac{9}{4}\left[\frac{224}{405}, \frac{458}{405}\right] = \left[\frac{56}{45}, \frac{229}{90}\right]. $$

We now evaluate the remaining integral appearing in (3.1):

$$ \int_0^1 \lambda^{\alpha-1}\,{}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_2}\Big)\,\mathcal{F}(\zeta_1 + \lambda\Delta)\,d\lambda. $$

Using the chosen parameters, this becomes

$$ \int_0^1 \lambda\left(1 - \frac{2\lambda}{3}\right)\mathcal{F}(1+2\lambda)\,d\lambda. $$

Since

$$ \mathcal{F}(1+2\lambda) = \left[(1+2\lambda)^2 + 1,\ (1+2\lambda)^3 + 2\right], $$

we obtain

$$ \int_0^1 \lambda\left(1 - \frac{2\lambda}{3}\right)\mathcal{F}(1+2\lambda)\,d\lambda = \left[\frac{76}{45}, \frac{71}{18}\right]. $$

Hence, the right-hand side of (3.1) becomes

$$ \begin{aligned} &\frac{1}{2}\left[\frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^\alpha}\mathcal{I}_{\zeta_1^+}^{\alpha,\beta,\gamma}\mathcal{F}(3) + \int_0^1 \lambda^{\alpha-1}\,{}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_2}\Big)\,\mathcal{F}(1+2\lambda)\,d\lambda\right] \\ &= \frac{1}{2}\left[\frac{56}{45} + \frac{76}{45}, \frac{229}{90} + \frac{71}{18}\right]. \end{aligned} $$

Therefore,

$$ \left[\frac{25}{18}, \frac{25}{9}\right] \preceq_{\mathrm{cr}} \left[\frac{22}{15}, \frac{146}{45}\right], $$

so inequality (3.1) is verified.

We next turn to (3.2). First we evaluate the coefficient on its left-hand side:

$$ \int_0^1 \lambda^{\alpha-1}(\zeta_1 + \lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta}{\zeta_1 + \lambda\Delta}\Big)\,d\lambda $$

$$= \int_0^1 \lambda(1+2\lambda)^{-2}\,{}_2F_1\left(2,-1;2;\frac{2\lambda}{1+2\lambda}\right)\,d\lambda.$$

Since

$${}_2F_1\left(2,-1;2;\frac{2\lambda}{1+2\lambda}\right) = 1-\frac{2\lambda}{1+2\lambda} = \frac{1}{1+2\lambda},$$

it follows that

$$\int_0^1 \lambda(1+2\lambda)^{-2}\,{}_2F_1\left(2,-1;2;\frac{2\lambda}{1+2\lambda}\right)\,d\lambda = \int_0^1 \frac{\lambda}{(1+2\lambda)^3}\,d\lambda = \frac{1}{18}.$$

Hence,

$$\left(\int_0^1 \lambda^{\alpha-1}(\zeta_1+\lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right)\,d\lambda\right)\mathcal{F}(2) = \frac{1}{18}[5,10] = \left[\frac{5}{18},\frac{5}{9}\right].$$

Now we compute the right-sided Saigo fractional integral at $\zeta_1 = 1$:

$$\mathcal{I}_{3^-}^{2,0,1}\mathcal{F}(1) = \frac{1}{\Gamma(2)}\int_1^3 (u-1)\,u^{-2}\,{}_2F_1\left(2,-1;2;1-\frac{1}{u}\right)\mathcal{F}(u)\,du.$$

Since $\Gamma(2) = 1$ and

$${}_2F_1\left(2,-1;2;1-\frac{1}{u}\right) = 1-\left(1-\frac{1}{u}\right) = \frac{1}{u},$$

we obtain

$$\mathcal{I}_{3^-}^{2,0,1}\mathcal{F}(1) = \int_1^3 \frac{u-1}{u^3}\mathcal{F}(u)\,du = \left[\frac{20}{9}-\ln 3,\ \frac{22}{9}\right].$$

Since

$$\frac{\Gamma(\alpha)}{\Delta^\alpha} = \frac{\Gamma(2)}{2^2} = \frac{1}{4},$$

it follows that

$$\frac{\Gamma(\alpha)}{\Delta^\alpha}\mathcal{I}_{3^-}^{2,0,1}\mathcal{F}(1) = \left[\frac{5}{9}-\frac{1}{4}\ln 3,\ \frac{11}{18}\right].$$

We now evaluate the remaining integral in (3.2):

$$\begin{aligned}
&\int_0^1 \lambda^{\alpha-1}(\zeta_1+\lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right)\mathcal{F}(\zeta_2-\lambda\Delta)\,d\lambda\\
&= \int_0^1 \frac{\lambda}{(1+2\lambda)^3}\mathcal{F}(3-2\lambda)\,d\lambda\\
&= \left[\frac{25}{9}-\frac{9}{4}\ln 3,\ \frac{103}{6}-15\ln 3\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\left[\int_0^1 \lambda^{\alpha-1}(\zeta_1+\lambda\Delta)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta}{\zeta_1+\lambda\Delta}\right)\mathcal{F}(\zeta_2-\lambda\Delta)\,d\lambda + \frac{\Gamma(\alpha)}{\Delta^\alpha}\mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\mathcal{F}(1)\right]\\
&= \frac{1}{2}\left[\frac{25}{9}-\frac{9}{4}\ln 3+\frac{5}{9}-\frac{1}{4}\ln 3,\ \frac{103}{6}-15\ln 3+\frac{11}{18}\right]
\end{aligned}$$

$$= \left[\frac{5}{3} - \frac{5}{4}\ln 3,\ \frac{80}{9} - \frac{15}{2}\ln 3\right].$$

Thus,

$$\left[\frac{5}{18}, \frac{5}{9}\right] \preceq_{\mathrm{cr}} \left[\frac{5}{3} - \frac{5}{4}\ln 3,\ \frac{80}{9} - \frac{15}{2}\ln 3\right] \approx [0.2934,\ 0.6493],$$

and therefore (3.2) is also verified.

Finally, combining the above two estimates, we conclude that the direct-form midpoint bounds in Theorem 3.1 hold for this example.

As shown in Theorem 3.1, we have established Hermite–Hadamard-type relations for interval-valued convex functions via Saigo fractional integrals. This result provides a generalized fractional framework involving hypergeometric kernels, from which several known inequalities can be recovered under suitable choices of the parameters and function classes. Table 1 summarizes some important reductions and related refinements that follow from our main theorem.

Table 1. Special cases recovered from Theorem 3.1.

Parameter Setting	Recovered Result	Reference
$\beta = -\alpha,\ \gamma = 0,\ \underline{\mathcal{F}} = \overline{\mathcal{F}}$	Riemann–Liouville fractional Hermite–Hadamard inequality	[40]
$\beta = -\alpha,\ \gamma = 0,\ \alpha = 1,\ \underline{\mathcal{F}} = \overline{\mathcal{F}}$	Classical Hermite–Hadamard inequality	[21]
$\alpha = 1,\ \beta = 0,\ \gamma = 0,$ interval-valued $\mathcal{F}$	Interval Hermite–Hadamard inequality in the non-fractional case	[14]
$\beta = -\alpha,\ \gamma = 0,$ interval-valued $\mathcal{F}$	Interval-valued Riemann–Liouville Hermite–Hadamard inequality	[38]
$\alpha = 1,\ \beta = -1,\ \gamma = 0,$ interval-valued $\mathcal{F}$	Classical interval Hermite–Hadamard inequality	[36]
$\beta = 0,\ \gamma \neq 0,\ \underline{\mathcal{F}} = \overline{\mathcal{F}}$	Erdélyi–Kober-type fractional Hermite–Hadamard inequality	[22]

Theorem 3.2. *Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}^{+}$ be an interval-valued convex mapping defined by*

$$\mathcal{F}(y) = \left[\underline{\mathcal{F}}(y), \overline{\mathcal{F}}(y)\right], \qquad y \in [\zeta_1, \zeta_2].$$

Let $\mathcal{G} : [\zeta_1, \zeta_2] \to \mathcal{R}$ be a nonnegative integrable function satisfying

$$\mathcal{G}(\zeta_1 + \zeta_2 - x) = \mathcal{G}(x), \qquad x \in [\zeta_1, \zeta_2].$$

Let $\alpha > 0$ and $\beta, \gamma \in \mathcal{R}$ satisfy (H)*, and write*

$$\Delta := \zeta_2 - \zeta_1.$$

Then

$$\left(\int_0^1 \eta^{\alpha-1}\, {}_2F_1\left(\alpha + \beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\right) \mathcal{G}(\zeta_2 - \eta\Delta)\, d\eta\right) \mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right)$$

$$\preceq_{\mathrm{cr}} \frac{1}{2}\Bigg[\frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^{\alpha}}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2) + \int_0^1 \eta^{\alpha-1}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\zeta_1+\eta\Delta)\,d\eta\Bigg], \tag{3.5}$$

and

$$\begin{aligned}&\bigg(\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\mathcal{G}(\zeta_1+\eta\Delta)\,d\eta\bigg)\mathcal{F}\bigg(\frac{\zeta_1+\zeta_2}{2}\bigg)\\ &\preceq_{\mathrm{cr}} \frac{1}{2}\Bigg[\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\mathcal{G}(\zeta_1+\eta\Delta)\,\mathcal{F}(\zeta_2-\eta\Delta)\,d\eta\\ &\quad+\frac{\Gamma(\alpha)}{\Delta^{\alpha}}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_1)\Bigg].\end{aligned} \tag{3.6}$$

Consequently,

$$\begin{aligned}&\Bigg[\int_0^1 \eta^{\alpha-1}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\mathcal{G}(\zeta_2-\eta\Delta)\,d\eta\\ &\quad+\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\mathcal{G}(\zeta_1+\eta\Delta)\,d\eta\Bigg]\mathcal{F}\bigg(\frac{\zeta_1+\zeta_2}{2}\bigg)\\ &\preceq_{\mathrm{cr}} \frac{1}{2}\Bigg[\frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^{\alpha}}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2)+\frac{\Gamma(\alpha)}{\Delta^{\alpha}}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_1)\\ &\quad+\int_0^1 \eta^{\alpha-1}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\zeta_1+\eta\Delta)\,d\eta\\ &\quad+\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\mathcal{G}(\zeta_1+\eta\Delta)\,\mathcal{F}(\zeta_2-\eta\Delta)\,d\eta\Bigg].\end{aligned} \tag{3.7}$$

Moreover,

$$\begin{aligned}&\int_0^1 \eta^{\alpha-1}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\zeta_1+\eta\Delta)\,d\eta\\ &\quad+\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\mathcal{G}(\zeta_1+\eta\Delta)\,\mathcal{F}(\zeta_2-\eta\Delta)\,d\eta\\ &\preceq_{\mathrm{cr}} \Bigg[\int_0^1 (1-\eta)\eta^{\alpha-1}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\mathcal{G}(\zeta_2-\eta\Delta)\,d\eta\\ &\quad+\int_0^1 \eta^{\alpha}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\mathcal{G}(\zeta_1+\eta\Delta)\,d\eta\Bigg]\mathcal{F}(\zeta_1)\\ &\quad+\Bigg[\int_0^1 \eta^{\alpha}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\mathcal{G}(\zeta_2-\eta\Delta)\,d\eta\\ &\quad+\int_0^1 (1-\eta)\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\mathcal{G}(\zeta_1+\eta\Delta)\,d\eta\Bigg]\mathcal{F}(\zeta_2).\end{aligned} \tag{3.8}$$

Proof. Since $\mathcal{F}$ is interval-valued convex on $[\zeta_1, \zeta_2]$, for every $\eta \in [0,1]$ we have

$$\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right) = \mathcal{F}\left(\frac{\big(\eta\zeta_1+(1-\eta)\zeta_2\big)+\big((1-\eta)\zeta_1+\eta\zeta_2\big)}{2}\right),$$

and therefore

$$\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right) \preceq_{\mathfrak{cr}} \frac{1}{2}\Big[\mathcal{F}\big(\eta\zeta_1+(1-\eta)\zeta_2\big)+\mathcal{F}\big((1-\eta)\zeta_1+\eta\zeta_2\big)\Big].$$

Multiplying this inequality by the nonnegative function

$$\eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)$$

and integrating over $[0,1]$, we obtain

$$\begin{aligned}
&\left(\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,d\eta\right)\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right)\\
\preceq_{\mathfrak{cr}}\ &\frac{1}{2}\Bigg[\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\eta\zeta_1+(1-\eta)\zeta_2)\,d\eta\\
&+\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}((1-\eta)\zeta_1+\eta\zeta_2)\,d\eta\Bigg].
\end{aligned}$$

Recalling that

$$\eta\zeta_1+(1-\eta)\zeta_2=\zeta_2-\eta\Delta, \qquad (1-\eta)\zeta_1+\eta\zeta_2=\zeta_1+\eta\Delta,$$

the preceding estimate becomes

$$\begin{aligned}
&\left(\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,d\eta\right)\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right)\\
\preceq_{\mathfrak{cr}}\ &\frac{1}{2}\Bigg[\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\zeta_2-\eta\Delta)\,d\eta\\
&+\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\zeta_1+\eta\Delta)\,d\eta\Bigg].
\end{aligned}$$

For the first integral on the right-hand side, use the substitution

$$u=\zeta_2-\eta\Delta, \qquad \eta=\frac{\zeta_2-u}{\Delta}, \qquad d\eta=-\frac{1}{\Delta}\,du.$$

When $\eta=0$, we have $u=\zeta_2$, and when $\eta=1$, we have $u=\zeta_1$. Hence,

$$\begin{aligned}
&\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\zeta_2-\eta\Delta)\,d\eta\\
&=\frac{1}{\Delta^{\alpha}}\int_{\zeta_1}^{\zeta_2}(\zeta_2-u)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{u}{\zeta_2}\right)\mathcal{G}(u)\mathcal{F}(u)\,du.
\end{aligned}$$

By recalling the definition of the left-sided Saigo fractional integral at ζ_2, we find that

$$\begin{aligned}&\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\zeta_2-\eta\Delta)\,d\eta\\&=\frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^\alpha}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2).\end{aligned}$$

Substituting this into the previous estimate yields (3.5).

Similarly, multiplying the same convexity inequality by the nonnegative function

$$\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\right)\mathcal{G}(\zeta_1+\eta\Delta)$$

and integrating over $[0,1]$, we obtain

$$\begin{aligned}&\left(\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\right)\mathcal{G}(\zeta_1+\eta\Delta)\,d\eta\right)\mathcal{F}\left(\frac{\zeta_1+\zeta_2}{2}\right)\\&\preceq_{\mathrm{cr}}\frac{1}{2}\Bigg[\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\right)\mathcal{G}(\zeta_1+\eta\Delta)\,\mathcal{F}(\zeta_2-\eta\Delta)\,d\eta\\&\quad+\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\right)\mathcal{G}(\zeta_1+\eta\Delta)\,\mathcal{F}(\zeta_1+\eta\Delta)\,d\eta\Bigg].\end{aligned}$$

Using the substitution

$$v=\zeta_1+\eta\Delta,\qquad \eta=\frac{v-\zeta_1}{\Delta},\qquad d\eta=\frac{1}{\Delta}\,dv,$$

we obtain

$$\begin{aligned}&\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\right)\mathcal{G}(\zeta_1+\eta\Delta)\,\mathcal{F}(\zeta_1+\eta\Delta)\,d\eta\\&=\frac{1}{\Delta^\alpha}\int_{\zeta_1}^{\zeta_2}(v-\zeta_1)^{\alpha-1}v^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{\zeta_1}{v}\right)\mathcal{G}(v)\mathcal{F}(v)\,dv.\end{aligned}$$

Recalling the definition of the right-sided Saigo fractional integral at ζ_1, it follows that

$$\begin{aligned}&\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\right)\mathcal{G}(\zeta_1+\eta\Delta)\,\mathcal{F}(\zeta_1+\eta\Delta)\,d\eta\\&=\frac{\Gamma(\alpha)}{\Delta^\alpha}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_1).\end{aligned}$$

Substituting this into the preceding estimate yields (3.6).

Adding (3.5) and (3.6), we directly obtain (3.7).

To establish (3.8), we again use convexity. Since

$$\mathcal{F}(\zeta_1+\eta\Delta)=\mathcal{F}((1-\eta)\zeta_1+\eta\zeta_2)\preceq_{\mathrm{cr}}(1-\eta)\mathcal{F}(\zeta_1)+\eta\mathcal{F}(\zeta_2),$$

and

$$\mathcal{F}(\zeta_2-\eta\Delta)=\mathcal{F}(\eta\zeta_1+(1-\eta)\zeta_2)\preceq_{\mathrm{cr}}\eta\mathcal{F}(\zeta_1)+(1-\eta)\mathcal{F}(\zeta_2),$$

we multiply the first inequality by

$$\eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)$$

and the second by

$$\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\right)\mathcal{G}(\zeta_1+\eta\Delta),$$

then integrate over $[0,1]$. This gives

$$\begin{aligned}
&\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\zeta_1+\eta\Delta)\,d\eta\\
&+\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\right)\mathcal{G}(\zeta_1+\eta\Delta)\,\mathcal{F}(\zeta_2-\eta\Delta)\,d\eta\\
\preceq_{\mathfrak{cr}}&\left[\int_0^1 (1-\eta)\eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,d\eta\right.\\
&\left.+\int_0^1 \eta^{\alpha}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\right)\mathcal{G}(\zeta_1+\eta\Delta)\,d\eta\right]\mathcal{F}(\zeta_1)\\
&+\left[\int_0^1 \eta^{\alpha}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,d\eta\right.\\
&\left.+\int_0^1 (1-\eta)\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\right)\mathcal{G}(\zeta_1+\eta\Delta)\,d\eta\right]\mathcal{F}(\zeta_2),
\end{aligned}$$

which is precisely (3.8). This completes the proof. □

Remark 3.2. Theorem 3.2 establishes a weighted Hermite–Hadamard type inequality involving Saigo fractional integrals for interval-valued convex mappings. In particular, when $\mathcal{G}(x)\equiv 1$, Theorem 3.2 reduces to the corresponding midpoint-type Hermite–Hadamard inclusion obtained in Theorem 3.1. Furthermore, when $\beta=-\alpha$ and $\gamma=0$, the Saigo fractional integral operator reduces to the classical Riemann–Liouville fractional integral operator, and hence Theorem 3.2 yields the corresponding weighted Hermite–Hadamard type inequality in the Riemann–Liouville setting. The symmetry condition

$$\mathcal{G}(\zeta_1+\zeta_2-x)=\mathcal{G}(x),\qquad x\in[\zeta_1,\zeta_2],$$

ensures that the weight function contributes in a balanced way to the left- and right-sided fractional integral terms. Moreover, when $\underline{\mathcal{F}}=\overline{\mathcal{F}}$, Theorem 3.2 reduces to the corresponding real-valued weighted fractional inequality. Several important special cases are listed in Table 2.

Example 3.2. Consider the interval $[\zeta_1,\zeta_2]=[1,3]$ and choose

$$\alpha=1,\qquad \beta=0,\qquad \gamma=1.$$

Let

$$\mathcal{F}(x)=\left[x^2+1,\ x^3+2\right],\qquad \mathcal{G}(x)=(x-1)(3-x),\qquad x\in[1,3].$$

Table 2. Special cases of Theorem 3.2

Parameters	Result
$\mathcal{G}(x) \equiv 1$	Midpoint Hermite–Hadamard inclusion (Theorem 3.1)
$\beta = -\alpha$, $\gamma = 0$	Weighted Riemann–Liouville Hermite–Hadamard inequality
$\beta = -\alpha$, $\gamma = 0$, $\mathcal{G}(x) \equiv 1$	Riemann–Liouville Hermite–Hadamard inequality
$\underline{\mathcal{F}} = \overline{\mathcal{F}}$	Real-valued weighted Saigo inequality
$\underline{\mathcal{F}} = \overline{\mathcal{F}}$, $\beta = -\alpha$, $\gamma = 0$	Real-valued weighted Riemann–Liouville inequality
$\underline{\mathcal{F}} = \overline{\mathcal{F}}$, $\alpha = 1$, $\beta = -1$, $\gamma = 0$, $\mathcal{G}(x) \equiv 1$	Classical Hermite–Hadamard inequality

Clearly,

$$0 \le x^2 + 1 \le x^3 + 2, \qquad x \in [1, 3],$$

and both endpoint functions $x^2 + 1$ and $x^3 + 2$ are convex on $[1, 3]$. Moreover,

$$\mathcal{G}(1 + 3 - x) = \mathcal{G}(x),$$

so $\mathcal{G}$ is symmetric with respect to the midpoint 2. Since ${}_2F_1(1, -1; 1; z) = 1 - z > 0$ on $[0, 1)$, condition (H) holds. Hence, all the assumptions of Theorem 3.2 are satisfied.

Since

$$\Delta = \zeta_2 - \zeta_1 = 3 - 1 = 2,$$

the midpoint term is

$$\mathcal{F}\left(\frac{\zeta_1 + \zeta_2}{2}\right) = \mathcal{F}(2) = [5, 10].$$

Also, since

$${}_2F_1(1, -1; 1; z) = 1 - z,$$

we have

$${}_2F_1\left(1, -1; 1; \frac{2\eta}{3}\right) = 1 - \frac{2\eta}{3},$$

and

$${}_2F_1\left(1, -1; 1; \frac{2\eta}{1 + 2\eta}\right) = 1 - \frac{2\eta}{1 + 2\eta} = \frac{1}{1 + 2\eta}.$$

Verification of (3.5)**.** We have

$$\begin{aligned} &\int_0^1 \eta^{\alpha-1} {}_2F_1\left(\alpha + \beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\right) \mathcal{G}(\zeta_2 - \eta\Delta)\, d\eta \\ &= \int_0^1 {}_2F_1\left(1, -1; 1; \frac{2\eta}{3}\right) \mathcal{G}(3 - 2\eta)\, d\eta. \end{aligned}$$

Now

$$\mathcal{G}(3 - 2\eta) = (3 - 2\eta - 1)\big(3 - (3 - 2\eta)\big) = 4\eta(1 - \eta),$$

so

$$\begin{aligned}\int_0^1 {}_2F_1\left(1,-1;1;\frac{2\eta}{3}\right)\mathcal{G}(3-2\eta)\,d\eta &= \int_0^1 \left(1-\frac{2\eta}{3}\right)4\eta(1-\eta)\,d\eta\\ &= 4\int_0^1 \left(\eta-\frac{5}{3}\eta^2+\frac{2}{3}\eta^3\right)d\eta\\ &= 4\left(\frac{1}{2}-\frac{5}{9}+\frac{1}{6}\right)\\ &= \frac{4}{9}.\end{aligned}$$

Hence,

$$\left(\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,d\eta\right)\mathcal{F}(2)=\frac{4}{9}[5,10]=\left[\frac{20}{9},\frac{40}{9}\right].$$

Next, we evaluate

$$\frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^{\alpha}}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2).$$

Since $\Gamma(1)=1$, $\zeta_2=3$, and $\Delta=2$, the coefficient is

$$\frac{\Gamma(1)3^{1+0}}{2^1}=\frac{3}{2}.$$

By the definition of the left-sided Saigo fractional integral,

$$\mathcal{I}^{1,0,1}_{1^+}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(3)=\frac{1}{3}\int_1^3 {}_2F_1\Big(1,-1;1;1-\frac{u}{3}\Big)\,\mathcal{F}(u)\mathcal{G}(u)\,du.$$

Since

$${}_2F_1\Big(1,-1;1;1-\frac{u}{3}\Big)=\frac{u}{3},$$

we obtain

$$\mathcal{I}^{1,0,1}_{1^+}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(3)=\frac{1}{9}\int_1^3 u\,\mathcal{F}(u)\mathcal{G}(u)\,du.$$

For the lower endpoint,

$$u(u^2+1)(u-1)(3-u)=-u^5+4u^4-4u^3+4u^2-3u.$$

Hence,

$$\frac{1}{9}\int_1^3 u(u^2+1)(u-1)(3-u)\,du=\frac{224}{135}.$$

For the upper endpoint,

$$u(u^3+2)(u-1)(3-u)=-u^6+4u^5-3u^4-2u^3+8u^2-6u,$$

and therefore

$$\frac{1}{9}\int_1^3 u(u^3+2)(u-1)(3-u)\,du=\frac{3484}{945}.$$

Thus,

$$\mathcal{I}_{1^+}^{1,0,1}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(3) = \left[\frac{224}{135}, \frac{3484}{945}\right].$$

Multiplying by $\frac{3}{2}$, we obtain

$$\frac{\Gamma(1)3^1}{2}\mathcal{I}_{1^+}^{1,0,1}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(3) = \left[\frac{112}{45}, \frac{1742}{315}\right].$$

We now evaluate the remaining integral appearing in (3.5):

$$\begin{aligned}&\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\zeta_1+\eta\Delta)\,d\eta\\ &= \int_0^1 {}_2F_1\left(1, -1; 1; \frac{2\eta}{3}\right)\mathcal{G}(3-2\eta)\,\mathcal{F}(1+2\eta)\,d\eta.\end{aligned}$$

Using

$${}_2F_1\left(1, -1; 1; \frac{2\eta}{3}\right) = 1 - \frac{2\eta}{3}, \qquad \mathcal{G}(3-2\eta) = 4\eta(1-\eta),$$

and

$$\mathcal{F}(1+2\eta) = \big[(1+2\eta)^2+1,\ (1+2\eta)^3+2\big],$$

we compute the lower endpoint:

$$\int_0^1 \left(1-\frac{2\eta}{3}\right)4\eta(1-\eta)\big((1+2\eta)^2+1\big)\,d\eta = \frac{32}{15},$$

and the upper endpoint:

$$\int_0^1 \left(1-\frac{2\eta}{3}\right)4\eta(1-\eta)\big((1+2\eta)^3+2\big)\,d\eta = \frac{1394}{315}.$$

Therefore,

$$\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\right)\mathcal{G}(\zeta_2-\eta\Delta)\,\mathcal{F}(\zeta_1+\eta\Delta)\,d\eta = \left[\frac{32}{15}, \frac{1394}{315}\right].$$

Hence the right-hand side of (3.5) is

$$\frac{1}{2}\left(\left[\frac{112}{45}, \frac{1742}{315}\right] + \left[\frac{32}{15}, \frac{1394}{315}\right]\right) = \frac{1}{2}\left[\frac{208}{45}, \frac{448}{45}\right] = \left[\frac{104}{45}, \frac{224}{45}\right].$$

Thus,

$$\left[\frac{20}{9}, \frac{40}{9}\right] \preceq_{\mathrm{cr}} \left[\frac{104}{45}, \frac{224}{45}\right] \approx [2.3111,\ 4.9778],$$

since the centers satisfy $\frac{10}{3} < \frac{164}{45}$.

Verification of (3.6)**.** For the chosen parameters, the kernel of (3.6) is

$$\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\left(1, -1; 1; \frac{2\eta}{1+2\eta}\right) = \frac{1}{1+2\eta}\cdot\frac{1}{1+2\eta} = \frac{1}{(1+2\eta)^2},$$

and

$$\mathcal{G}(\zeta_1 + \eta\Delta) = \mathcal{G}(1 + 2\eta) = (2\eta)(2 - 2\eta) = 4\eta(1 - \eta).$$

Left-hand side. Using the substitution $t = 1 + 2\eta$ (so that $4\eta(1-\eta) = (t-1)(3-t)$ and $d\eta = \frac{1}{2}\,dt$), we get

$$\begin{aligned}\int_0^1 \frac{4\eta(1-\eta)}{(1+2\eta)^2}\,d\eta &= \frac{1}{2}\int_1^3 \frac{(t-1)(3-t)}{t^2}\,dt \\ &= \frac{1}{2}\int_1^3 \left(-1 + \frac{4}{t} - \frac{3}{t^2}\right) dt \\ &= \frac{1}{2}\left[-t + 4\ln t + \frac{3}{t}\right]_1^3 \\ &= 2\ln 3 - 2.\end{aligned}$$

Consequently, the left-hand side of (3.6) equals

$$(2\ln 3 - 2)\,\mathcal{F}(2) = (2\ln 3 - 2)[5, 10] = \big[10\ln 3 - 10,\ 20\ln 3 - 20\big] \approx [0.9861,\ 1.9722].$$

Right-sided Saigo integral. Since $\Gamma(1) = 1$ and ${}_2F_1\big(1, -1; 1; 1 - \frac{1}{u}\big) = \frac{1}{u}$,

$$\mathcal{I}_{3^-}^{1,0,1}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(1) = \int_1^3 u^{-1} \cdot \frac{1}{u}\,\mathcal{F}(u)\mathcal{G}(u)\,du = \int_1^3 \frac{\mathcal{F}(u)(u-1)(3-u)}{u^2}\,du.$$

For the lower endpoint,

$$\frac{(u^2+1)(u-1)(3-u)}{u^2} = -u^2 + 4u - 4 + \frac{4}{u} - \frac{3}{u^2},$$

so that

$$\int_1^3 \left(-u^2 + 4u - 4 + \frac{4}{u} - \frac{3}{u^2}\right) du = \left[-\frac{u^3}{3} + 2u^2 - 4u + 4\ln u + \frac{3}{u}\right]_1^3 = 4\ln 3 - \frac{8}{3}.$$

For the upper endpoint,

$$\frac{(u^3+2)(u-1)(3-u)}{u^2} = -u^3 + 4u^2 - 3u - 2 + \frac{8}{u} - \frac{6}{u^2},$$

so that

$$\int_1^3 \left(-u^3 + 4u^2 - 3u - 2 + \frac{8}{u} - \frac{6}{u^2}\right) du = \left[-\frac{u^4}{4} + \frac{4u^3}{3} - \frac{3u^2}{2} - 2u + 8\ln u + \frac{6}{u}\right]_1^3 = 8\ln 3 - \frac{16}{3}.$$

Hence

$$\mathcal{I}_{3^-}^{1,0,1}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(1) = \left[4\ln 3 - \frac{8}{3},\ 8\ln 3 - \frac{16}{3}\right],$$

and, with the coefficient $\frac{\Gamma(1)}{\Delta} = \frac{1}{2}$,

$$\frac{\Gamma(\alpha)}{\Delta^{\alpha}}\,\mathcal{I}_{3^-}^{1,0,1}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(1) = \left[2\ln 3 - \frac{4}{3},\ 4\ln 3 - \frac{8}{3}\right].$$

Remaining integral. With the same substitution $t = 1 + 2\eta$ we have $3 - 2\eta = 4 - t$, and therefore

$$\int_0^1 \frac{4\eta(1-\eta)}{(1+2\eta)^2}\,\mathcal{F}(3-2\eta)\,d\eta = \frac{1}{2}\int_1^3 \frac{(t-1)(3-t)}{t^2}\,\mathcal{F}(4-t)\,dt.$$

Expanding the rational integrands,

$$\frac{(t-1)(3-t)\big((4-t)^2+1\big)}{t^2} = -t^2 + 12t - 52 + \frac{92}{t} - \frac{51}{t^2},$$
$$\frac{(t-1)(3-t)\big((4-t)^3+2\big)}{t^2} = t^3 - 16t^2 + 95t - 252 + \frac{408}{t} - \frac{198}{t^2},$$

and integrating over $[1,3]$ gives

$$\frac{1}{2}\int_1^3 \frac{(t-1)(3-t)}{t^2}\,\mathcal{F}(4-t)\,dt = \left[46\ln 3 - \frac{148}{3},\ 204\ln 3 - \frac{664}{3}\right] \approx [1.2028,\ 2.7836].$$

Right-hand side. Collecting the above,

$$\begin{aligned}
&\frac{1}{2}\left(\left[46\ln 3 - \frac{148}{3},\ 204\ln 3 - \frac{664}{3}\right] + \left[2\ln 3 - \frac{4}{3},\ 4\ln 3 - \frac{8}{3}\right]\right)\\
&= \frac{1}{2}\left[48\ln 3 - \frac{152}{3},\ 208\ln 3 - 224\right]\\
&= \left[24\ln 3 - \frac{76}{3},\ 104\ln 3 - 112\right]\\
&\approx [1.0334,\ 2.2557].
\end{aligned}$$

Therefore,

$$\big[10\ln 3 - 10,\ 20\ln 3 - 20\big] \preceq_{\mathrm{cr}} \left[24\ln 3 - \frac{76}{3},\ 104\ln 3 - 112\right],$$

because the center of the left-hand interval ($15\ln 3 - 15 \approx 1.4792$) is smaller than the center of the right-hand interval ($64\ln 3 - \frac{206}{3} \approx 1.6445$). Thus (3.6) is verified, and adding the two verified inequalities also confirms (3.7) for this example.

Theorem 3.3. *Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F}, \mathcal{G} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}^{+}$ be interval-valued mappings defined by*

$$\mathcal{F}(y) = \big[\underline{\mathcal{F}}(y), \overline{\mathcal{F}}(y)\big], \qquad \mathcal{G}(y) = \big[\underline{\mathcal{G}}(y), \overline{\mathcal{G}}(y)\big].$$

Assume that $\mathcal{F}$ and $\mathcal{G}$ are interval-valued convex on $[\zeta_1, \zeta_2]$ in the sense of Definition 2.6.

Let $\alpha > 0$, $\beta, \gamma \in \mathcal{R}$ satisfy (H)*, and write*

$$\Delta := \zeta_2 - \zeta_1.$$

Then

$$\begin{aligned}
&\frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^{\alpha}}\mathcal{I}_{\zeta_1^+}^{\alpha,\beta,\gamma}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2)\\
&\preceq_{\mathrm{cr}} \left(\int_0^1 \eta^{\alpha-1}\big[(1-\eta)^2+\eta^2\big]\,{}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\right)\,d\eta\right)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1) + \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\Big]
\end{aligned}$$

$$+\left(2\int_0^1 \eta^{\alpha}(1-\eta)\,{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\Big)\,d\eta\right)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big], \tag{3.9}$$

and

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{\Delta^{\alpha}}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_1)\\
&\preceq_{\mathrm{cr}}\left(\int_0^1 \eta^{\alpha-1}\big[(1-\eta)^2+\eta^2\big](\zeta_1+\eta\Delta)^{-\alpha-\beta}\,{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\Big)\,d\eta\right)\\
&\quad\times\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\Big]\\
&\quad+\left(2\int_0^1 \eta^{\alpha}(1-\eta)(\zeta_1+\eta\Delta)^{-\alpha-\beta}\,{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\Big)\,d\eta\right)\\
&\quad\times\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big].
\end{aligned} \tag{3.10}$$

Consequently,

$$\begin{aligned}
&\frac{1}{2}\left[\frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^{\alpha}}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2)+\frac{\Gamma(\alpha)}{\Delta^{\alpha}}\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_2^-}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_1)\right]\\
&\preceq_{\mathrm{cr}}\frac{1}{2}\left[\int_0^1 \eta^{\alpha-1}\big[(1-\eta)^2+\eta^2\big]\,{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\Big)\,d\eta\right.\\
&\quad\left.+\int_0^1 \eta^{\alpha-1}\big[(1-\eta)^2+\eta^2\big](\zeta_1+\eta\Delta)^{-\alpha-\beta}\,{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\Big)\,d\eta\right]\\
&\quad\times\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\Big]\\
&\quad+\frac{1}{2}\left[2\int_0^1 \eta^{\alpha}(1-\eta)\,{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\Big)\,d\eta\right.\\
&\quad\left.+2\int_0^1 \eta^{\alpha}(1-\eta)(\zeta_1+\eta\Delta)^{-\alpha-\beta}\,{}_2F_1\Big(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\Big)\,d\eta\right]\\
&\quad\times\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big].
\end{aligned} \tag{3.11}$$

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ are interval-valued convex on $[\zeta_1,\zeta_2]$ (Definition 2.6), for every $\eta\in[0,1]$ we have the endpoint-wise inequalities

$$\mathcal{F}\big((1-\eta)\zeta_1+\eta\zeta_2\big)\le_{\mathrm{LU}}(1-\eta)\mathcal{F}(\zeta_1)+\eta\mathcal{F}(\zeta_2),$$

and

$$\mathcal{G}\big((1-\eta)\zeta_1+\eta\zeta_2\big)\le_{\mathrm{LU}}(1-\eta)\mathcal{G}(\zeta_1)+\eta\mathcal{G}(\zeta_2).$$

All intervals involved are nonnegative. Hence, by Lemma 2.1(ii), the two inequalities may be multiplied, and expanding the product of the right-hand sides (which, for nonnegative intervals and nonnegative scalars, is computed endpoint-wise) we get

$$\begin{aligned}
&\mathcal{F}\big((1-\eta)\zeta_1+\eta\zeta_2\big)\mathcal{G}\big((1-\eta)\zeta_1+\eta\zeta_2\big)\\
&\preceq_{\mathrm{cr}}(1-\eta)^2\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\eta^2\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)
\end{aligned}$$

$$+\eta(1-\eta)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big], \tag{3.12}$$

where in fact the inequality holds in the stronger sense $\leq_{\mathrm{LU}}$, and $\preceq_{\mathrm{cr}}$ follows from Lemma 2.1(i).

Similarly,

$$\mathcal{F}\big(\eta\zeta_1+(1-\eta)\zeta_2\big) \leq_{\mathrm{LU}} \eta\mathcal{F}(\zeta_1)+(1-\eta)\mathcal{F}(\zeta_2),$$

and

$$\mathcal{G}\big(\eta\zeta_1+(1-\eta)\zeta_2\big) \leq_{\mathrm{LU}} \eta\mathcal{G}(\zeta_1)+(1-\eta)\mathcal{G}(\zeta_2),$$

so that, again by Lemma 2.1(ii) and (i),

$$\begin{aligned}
&\mathcal{F}\big(\eta\zeta_1+(1-\eta)\zeta_2\big)\mathcal{G}\big(\eta\zeta_1+(1-\eta)\zeta_2\big)\\
&\preceq_{\mathrm{cr}} \eta^2\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+(1-\eta)^2\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\\
&\quad+\eta(1-\eta)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big].
\end{aligned} \tag{3.13}$$

Adding (3.12) and (3.13), we obtain

$$\begin{aligned}
&\mathcal{F}\big(\eta\zeta_1+(1-\eta)\zeta_2\big)\mathcal{G}\big(\eta\zeta_1+(1-\eta)\zeta_2\big)\\
&+\mathcal{F}\big((1-\eta)\zeta_1+\eta\zeta_2\big)\mathcal{G}\big((1-\eta)\zeta_1+\eta\zeta_2\big)\\
&\preceq_{\mathrm{cr}} \big[(1-\eta)^2+\eta^2\big]\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\Big]\\
&\quad+2\eta(1-\eta)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big].
\end{aligned} \tag{3.14}$$

Multiplying both sides of (3.14) by the nonnegative kernel

$$\eta^{\alpha-1}\,{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)$$

and integrating over $[0,1]$, we get

$$\begin{aligned}
&\int_0^1 \eta^{\alpha-1}\,{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\,\mathcal{F}\big(\eta\zeta_1+(1-\eta)\zeta_2\big)\mathcal{G}\big(\eta\zeta_1+(1-\eta)\zeta_2\big)\,d\eta\\
&+\int_0^1 \eta^{\alpha-1}\,{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\,\mathcal{F}\big((1-\eta)\zeta_1+\eta\zeta_2\big)\mathcal{G}\big((1-\eta)\zeta_1+\eta\zeta_2\big)\,d\eta\\
&\preceq_{\mathrm{cr}}\bigg(\int_0^1 \eta^{\alpha-1}\big[(1-\eta)^2+\eta^2\big]\,{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\,d\eta\bigg)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\Big]\\
&+\bigg(2\int_0^1 \eta^{\alpha}(1-\eta)\,{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\,d\eta\bigg)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big].
\end{aligned} \tag{3.15}$$

Recalling that

$$\eta\zeta_1+(1-\eta)\zeta_2=\zeta_2-\eta\Delta,$$

we use the substitution

$$u=\zeta_2-\eta\Delta,\qquad \eta=\frac{\zeta_2-u}{\Delta},\qquad d\eta=-\frac{1}{\Delta}\,du.$$

Then

$$\int_0^1 \eta^{\alpha-1}\,{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\,\mathcal{F}(\zeta_2-\eta\Delta)\mathcal{G}(\zeta_2-\eta\Delta)\,d\eta$$

$$= \frac{1}{\Delta^{\alpha}} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - u)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1 - \frac{u}{\zeta_2}\right) \mathcal{F}(u)\mathcal{G}(u)\, du.$$

By the definition of the left-sided Saigo fractional integral,

$$\mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2) = \frac{\zeta_2^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{\zeta_1}^{\zeta_2} (\zeta_2 - u)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1 - \frac{u}{\zeta_2}\right) \mathcal{F}(u)\mathcal{G}(u)\, du,$$

and hence

$$\begin{aligned} &\int_0^1 \eta^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\right) \mathcal{F}(\zeta_2 - \eta\Delta)\mathcal{G}(\zeta_2 - \eta\Delta)\, d\eta \\ &= \frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^{\alpha}} \mathcal{I}^{\alpha,\beta,\gamma}_{\zeta_1^+}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2). \end{aligned}$$

Substituting this into (3.15), and observing that the remaining integral on the left-hand side is a nonnegative interval-valued quantity (so that dropping it preserves the order), we obtain (3.9).

Similarly, multiplying (3.14) by the nonnegative kernel

$$\eta^{\alpha-1}(\zeta_1 + \eta\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1 + \eta\Delta}\right)$$

and integrating over $[0,1]$, we have

$$\begin{aligned} &\int_0^1 \eta^{\alpha-1}(\zeta_1 + \eta\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1 + \eta\Delta}\right) \mathcal{F}\big(\eta\zeta_1 + (1-\eta)\zeta_2\big) \\ &\times \mathcal{G}\big(\eta\zeta_1 + (1-\eta)\zeta_2\big)\, d\eta \\ &+ \int_0^1 \eta^{\alpha-1}(\zeta_1 + \eta\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1 + \eta\Delta}\right) \mathcal{F}\big((1-\eta)\zeta_1 + \eta\zeta_2\big) \\ &\times \mathcal{G}\big((1-\eta)\zeta_1 + \eta\zeta_2\big)\, d\eta \\ \preceq_{\mathrm{cr}} &\left(\int_0^1 \eta^{\alpha-1}\big[(1-\eta)^2 + \eta^2\big](\zeta_1 + \eta\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1 + \eta\Delta}\right) d\eta\right) \\ &\times \Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1) + \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\Big] \\ &+ \left(2\int_0^1 \eta^{\alpha}(1-\eta)(\zeta_1 + \eta\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1 + \eta\Delta}\right) d\eta\right) \\ &\times \Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2) + \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big]. \end{aligned} \tag{3.16}$$

Recalling that

$$(1-\eta)\zeta_1 + \eta\zeta_2 = \zeta_1 + \eta\Delta,$$

we use the substitution

$$v = \zeta_1 + \eta\Delta, \qquad \eta = \frac{v - \zeta_1}{\Delta}, \qquad d\eta = \frac{1}{\Delta}\, dv.$$

Then

$$\int_0^1 \eta^{\alpha-1}(\zeta_1 + \eta\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1 + \eta\Delta}\right) \mathcal{F}(\zeta_1 + \eta\Delta)\mathcal{G}(\zeta_1 + \eta\Delta)\, d\eta$$

$$= \frac{1}{\Delta^{\alpha}} \int_{\zeta_1}^{\zeta_2} (v-\zeta_1)^{\alpha-1} v^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{\zeta_1}{v}\right) \mathcal{F}(v)\mathcal{G}(v)\, dv.$$

By the definition of the right-sided Saigo fractional integral,

$$\mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_1) = \frac{1}{\Gamma(\alpha)} \int_{\zeta_1}^{\zeta_2} (v-\zeta_1)^{\alpha-1} v^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{\zeta_1}{v}\right) \mathcal{F}(v)\mathcal{G}(v)\, dv,$$

which gives

$$\int_0^1 \eta^{\alpha-1} (\zeta_1+\eta\Delta)^{-\alpha-\beta}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1+\eta\Delta}\right) \mathcal{F}(\zeta_1+\eta\Delta)\mathcal{G}(\zeta_1+\eta\Delta)\, d\eta$$
$$= \frac{\Gamma(\alpha)}{\Delta^{\alpha}} \mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_1).$$

Substituting this into (3.16), and dropping the remaining nonnegative interval-valued term on the left-hand side, we obtain (3.10).

Finally, adding (3.9) and (3.10), then dividing by 2, yields (3.11). This completes the proof. □

Example 3.3. Let $[\zeta_1, \zeta_2] = [1, 2]$, and choose

$$\alpha = 1, \qquad \beta = 0, \qquad \gamma = 0.$$

Define the interval-valued mappings

$$\mathcal{F}(y) = \big[y,\, 2y\big], \qquad \mathcal{G}(y) = \big[y+1,\, y+2\big], \qquad y \in [1, 2].$$

Then

$$\underline{\mathcal{F}}(y) = y, \qquad \overline{\mathcal{F}}(y) = 2y,$$

and

$$\underline{\mathcal{G}}(y) = y+1, \qquad \overline{\mathcal{G}}(y) = y+2.$$

Since all endpoint functions are linear on $[1, 2]$, they are convex. Hence $\mathcal{F}$ and $\mathcal{G}$ are interval-valued convex on $[1, 2]$, and since ${}_2F_1(1, 0; 1; z) = 1$, condition (H) holds; so all assumptions of Theorem 3.3 are satisfied.

Also,

$$\Delta = \zeta_2 - \zeta_1 = 2 - 1 = 1.$$

At the endpoints we have

$$\mathcal{F}(1) = [1, 2], \qquad \mathcal{F}(2) = [2, 4],$$

and

$$\mathcal{G}(1) = [2, 3], \qquad \mathcal{G}(2) = [3, 4].$$

Thus

$$\mathcal{F}(1)\mathcal{G}(1) = [1, 2][2, 3] = [2, 6], \qquad \mathcal{F}(2)\mathcal{G}(2) = [2, 4][3, 4] = [6, 16],$$

so that

$$\mathcal{F}(1)\mathcal{G}(1) + \mathcal{F}(2)\mathcal{G}(2) = [2, 6] + [6, 16] = [8, 22].$$

Similarly,

$$\mathcal{F}(1)\mathcal{G}(2) = [1,2][3,4] = [3,8], \qquad \mathcal{F}(2)\mathcal{G}(1) = [2,4][2,3] = [4,12],$$

hence

$$\mathcal{F}(1)\mathcal{G}(2) + \mathcal{F}(2)\mathcal{G}(1) = [3,8] + [4,12] = [7,20].$$

Since all endpoint functions are positive on $[1,2]$, interval multiplication is componentwise, and therefore

$$\mathcal{F}(y)\mathcal{G}(y) = [y,2y][y+1,y+2] = \big[y(y+1),\, 2y(y+2)\big] = \big[y^2+y,\, 2y^2+4y\big].$$

We first compute the left-hand side of (3.9). Since

$${}_2F_1(1,0;1;z) = 1$$

for every admissible z, the left-sided Saigo fractional integral becomes

$$\mathcal{I}_{1^+}^{1,0,0}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(2) = \frac{2^{-1}}{\Gamma(1)}\int_1^2 (2-y)^0\, {}_2F_1\Big(1,0;1;1-\frac{y}{2}\Big)\,\big(\mathcal{F}(y)\mathcal{G}(y)\big)\,dy.$$

Since $\Gamma(1)=1$, $(2-y)^0=1$, and the hypergeometric term is equal to 1, this reduces to

$$\mathcal{I}_{1^+}^{1,0,0}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(2) = \frac{1}{2}\int_1^2 \big[y^2+y,\; 2y^2+4y\big]\,dy.$$

Hence

$$\mathcal{I}_{1^+}^{1,0,0}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(2) = \frac{1}{2}\left[\int_1^2 (y^2+y)\,dy,\; \int_1^2 (2y^2+4y)\,dy\right].$$

Now

$$\int_1^2 (y^2+y)\,dy = \left[\frac{y^3}{3}+\frac{y^2}{2}\right]_1^2 = \left(\frac{8}{3}+2\right) - \left(\frac{1}{3}+\frac{1}{2}\right) = \frac{23}{6},$$

and

$$\int_1^2 (2y^2+4y)\,dy = \left[\frac{2y^3}{3}+2y^2\right]_1^2 = \left(\frac{16}{3}+8\right) - \left(\frac{2}{3}+2\right) = \frac{32}{3}.$$

Therefore,

$$\mathcal{I}_{1^+}^{1,0,0}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(2) = \frac{1}{2}\left[\frac{23}{6},\frac{32}{3}\right] = \left[\frac{23}{12},\frac{16}{3}\right].$$

Multiplying by the coefficient appearing in (3.9), we obtain

$$\frac{\Gamma(1)\,2^{1+0}}{1^1}\mathcal{I}_{1^+}^{1,0,0}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(2) = 2\left[\frac{23}{12},\frac{16}{3}\right] = \left[\frac{23}{6},\frac{32}{3}\right].$$

We now compute the right-hand side of (3.9). Since $\alpha=1$, $\beta=0$, $\gamma=0$, $\Delta=1$, and $\zeta_2=2$, we have

$${}_2F_1\Big(1,0;1;\frac{\eta}{2}\Big) = 1.$$

Thus,

$$\int_0^1 \eta^{\alpha-1}\big[(1-\eta)^2+\eta^2\big]\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\,d\eta$$

$$= \int_0^1 \left[(1-\eta)^2 + \eta^2\right] d\eta = \int_0^1 (1 - 2\eta + 2\eta^2)\, d\eta = \left[\eta - \eta^2 + \frac{2}{3}\eta^3\right]_0^1 = \frac{2}{3},$$

and

$$2\int_0^1 \eta^{\alpha}(1-\eta)\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\right) d\eta$$
$$= 2\int_0^1 \eta(1-\eta)\, d\eta = 2\int_0^1 (\eta - \eta^2)\, d\eta = 2\left[\frac{\eta^2}{2} - \frac{\eta^3}{3}\right]_0^1 = \frac{1}{3}.$$

Hence the right-hand side of (3.9) becomes

$$\frac{2}{3}[8, 22] + \frac{1}{3}[7, 20] = \left[\frac{16}{3}, \frac{44}{3}\right] + \left[\frac{7}{3}, \frac{20}{3}\right] = \left[\frac{23}{3}, \frac{64}{3}\right].$$

Therefore,

$$\left[\frac{23}{6}, \frac{32}{3}\right] \preceq_{\mathrm{cr}} \left[\frac{23}{3}, \frac{64}{3}\right],$$

so (3.9) is verified.

Next, we verify (3.10). For the chosen parameters the right-sided kernel is $\eta^0(1+\eta)^{-1} \cdot 1 = (1+\eta)^{-1}$ and $\Gamma(1)/\Delta = 1$. By definition,

$$\mathcal{I}_{2^-}^{1,0,0}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(1) = \int_1^2 v^{-1}\left[v^2 + v,\ 2v^2 + 4v\right] dv = \left[\int_1^2 (v+1)\, dv,\ \int_1^2 (2v+4)\, dv\right] = \left[\frac{5}{2},\ 7\right].$$

Moreover,

$$\int_0^1 \frac{(1-\eta)^2 + \eta^2}{1+\eta}\, d\eta = \int_0^1 \left(2\eta - 4 + \frac{5}{1+\eta}\right) d\eta = 5\ln 2 - 3,$$
$$2\int_0^1 \frac{\eta(1-\eta)}{1+\eta}\, d\eta = 2\int_0^1 \left(2 - \eta - \frac{2}{1+\eta}\right) d\eta = 3 - 4\ln 2.$$

Hence the right-hand side of (3.10) is

$$(5\ln 2 - 3)[8, 22] + (3 - 4\ln 2)[7, 20] = \left[12\ln 2 - 3,\ 30\ln 2 - 6\right],$$

and

$$\left[\frac{5}{2},\ 7\right] \preceq_{\mathrm{cr}} \left[12\ln 2 - 3,\ 30\ln 2 - 6\right] \approx [5.3178,\ 14.7944],$$

so (3.10) is verified.

Finally, adding the two verified inequalities and dividing by 2, the combined inequality (3.11) reads

$$\left[\frac{19}{6}, \frac{53}{6}\right] \preceq_{\mathrm{cr}} \left[\frac{18\ln 2 + 7}{3}, \frac{45\ln 2 + 23}{3}\right],$$

which is therefore also verified for this example.

Theorem 3.4. *Let $\zeta_1, \zeta_2 \in \mathcal{R}$ with $0 < \zeta_1 < \zeta_2$, and let $\mathcal{F}, \mathcal{G} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}^{+}$ be interval-valued mappings, defined by*

$$\mathcal{F}(y) = \left[\underline{\mathcal{F}}(y), \overline{\mathcal{F}}(y)\right], \qquad \mathcal{G}(y) = \left[\underline{\mathcal{G}}(y), \overline{\mathcal{G}}(y)\right], \qquad y \in [\zeta_1, \zeta_2].$$

Assume that both $\mathcal{F}$ and $\mathcal{G}$ are interval-valued convex on $[\zeta_1, \zeta_2]$ in the sense of Definition 2.6.
Let $\alpha > 0$ and $\beta, \gamma \in \mathcal{R}$ satisfy (H)*, and put*

$$\Delta := \zeta_2 - \zeta_1, \qquad m := \frac{\zeta_1 + \zeta_2}{2}.$$

Then

$$\begin{aligned}
&2\Bigg[\int_0^1 \eta^{\alpha-1}\, {}_2F_1\bigg(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\bigg)\, d\eta \\
&+ \int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}\, {}_2F_1\bigg(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\, d\eta\Bigg]\mathcal{F}(m)\mathcal{G}(m) \\
\preceq_{\mathrm{cr}}\ & \frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{2\Delta^{\alpha}}\mathcal{I}_{\zeta_1^+}^{\alpha,\beta,\gamma}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2) + \frac{\Gamma(\alpha)}{2\Delta^{\alpha}}\mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_1) \\
&+ \frac{1}{2}\Bigg[\int_0^1 \Bigg((1-\eta^2)\eta^{\alpha-1}\, {}_2F_1\bigg(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\bigg) + (2\eta-\eta^2)\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta} \\
&\times {}_2F_1\bigg(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\Bigg)\, d\eta\Bigg]\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1) \\
&+ \frac{1}{2}\Bigg[\int_0^1 \Bigg((2\eta-\eta^2)\eta^{\alpha-1}\, {}_2F_1\bigg(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\bigg) + (1-\eta^2)\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta} \\
&\times {}_2F_1\bigg(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\Bigg)\, d\eta\Bigg]\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2) \\
&+ \frac{1}{2}\Bigg[\int_0^1 (1-\eta+\eta^2)\Bigg(\eta^{\alpha-1}\, {}_2F_1\bigg(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_2}\bigg) \\
&+ \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}\, {}_2F_1\bigg(\alpha+\beta, -\gamma; \alpha; \frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\Bigg)\, d\eta\Bigg] \\
&\times \Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2) + \mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big]. \qquad (3.17)
\end{aligned}$$

Proof. For $\eta \in [0,1]$, set

$$a_\eta := (1-\eta)\zeta_1 + \eta\zeta_2 = \zeta_1 + \eta\Delta, \qquad b_\eta := \eta\zeta_1 + (1-\eta)\zeta_2 = \zeta_2 - \eta\Delta.$$

Then

$$\frac{a_\eta + b_\eta}{2} = \frac{(1-\eta)\zeta_1 + \eta\zeta_2 + \eta\zeta_1 + (1-\eta)\zeta_2}{2} = \frac{\zeta_1+\zeta_2}{2} = m.$$

By convexity of $\mathcal{F}$ and $\mathcal{G}$ (Definition 2.6), we have the endpoint-wise inequalities

$$\mathcal{F}(m) = \mathcal{F}\bigg(\frac{a_\eta + b_\eta}{2}\bigg) \le_{\mathrm{LU}} \frac{\mathcal{F}(a_\eta) + \mathcal{F}(b_\eta)}{2},$$

and

$$\mathcal{G}(m) = \mathcal{G}\bigg(\frac{a_\eta + b_\eta}{2}\bigg) \le_{\mathrm{LU}} \frac{\mathcal{G}(a_\eta) + \mathcal{G}(b_\eta)}{2}.$$

Since all interval values are nonnegative, Lemma 2.1(ii) allows us to multiply these two inequalities, and Lemma 2.1(i) then yields

$$\begin{aligned}\mathcal{F}(m)\mathcal{G}(m) &\preceq_{\mathrm{cr}} \frac{1}{4}\Big[\mathcal{F}(a_\eta)+\mathcal{F}(b_\eta)\Big]\Big[\mathcal{G}(a_\eta)+\mathcal{G}(b_\eta)\Big]\\ &= \frac{1}{4}\Big[\mathcal{F}(a_\eta)\mathcal{G}(a_\eta)+\mathcal{F}(b_\eta)\mathcal{G}(b_\eta)\Big]\\ &\quad+\frac{1}{4}\Big[\mathcal{F}(a_\eta)\mathcal{G}(b_\eta)+\mathcal{F}(b_\eta)\mathcal{G}(a_\eta)\Big]. \end{aligned} \tag{3.18}$$

Here the distributive law used in the last step is valid because, for nonnegative intervals, all products are computed endpoint-wise.

Again by convexity (endpoint-wise) and Lemma 2.1(ii),

$$\mathcal{F}(a_\eta) \le_{\mathrm{LU}} (1-\eta)\mathcal{F}(\zeta_1)+\eta\mathcal{F}(\zeta_2), \qquad \mathcal{G}(a_\eta) \le_{\mathrm{LU}} (1-\eta)\mathcal{G}(\zeta_1)+\eta\mathcal{G}(\zeta_2).$$

Therefore,

$$\begin{aligned}\mathcal{F}(a_\eta)\mathcal{G}(a_\eta) &\preceq_{\mathrm{cr}} \Big[(1-\eta)\mathcal{F}(\zeta_1)+\eta\mathcal{F}(\zeta_2)\Big]\Big[(1-\eta)\mathcal{G}(\zeta_1)+\eta\mathcal{G}(\zeta_2)\Big]\\ &= (1-\eta)^2\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\eta^2\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\\ &\quad+\eta(1-\eta)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big].\end{aligned}$$

Likewise,

$$\begin{aligned}\mathcal{F}(b_\eta)\mathcal{G}(b_\eta) &\preceq_{\mathrm{cr}} \Big[\eta\mathcal{F}(\zeta_1)+(1-\eta)\mathcal{F}(\zeta_2)\Big]\Big[\eta\mathcal{G}(\zeta_1)+(1-\eta)\mathcal{G}(\zeta_2)\Big]\\ &= \eta^2\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+(1-\eta)^2\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\\ &\quad+\eta(1-\eta)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big].\end{aligned}$$

Similarly, for the cross terms,

$$\begin{aligned}\mathcal{F}(a_\eta)\mathcal{G}(b_\eta) &\preceq_{\mathrm{cr}} \Big[(1-\eta)\mathcal{F}(\zeta_1)+\eta\mathcal{F}(\zeta_2)\Big]\Big[\eta\mathcal{G}(\zeta_1)+(1-\eta)\mathcal{G}(\zeta_2)\Big]\\ &= \eta(1-\eta)\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\eta(1-\eta)\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\\ &\quad+(1-\eta)^2\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\eta^2\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1),\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}(b_\eta)\mathcal{G}(a_\eta) &\preceq_{\mathrm{cr}} \Big[\eta\mathcal{F}(\zeta_1)+(1-\eta)\mathcal{F}(\zeta_2)\Big]\Big[(1-\eta)\mathcal{G}(\zeta_1)+\eta\mathcal{G}(\zeta_2)\Big]\\ &= \eta(1-\eta)\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\eta(1-\eta)\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\\ &\quad+\eta^2\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+(1-\eta)^2\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1).\end{aligned}$$

Hence,

$$\begin{aligned}&\mathcal{F}(a_\eta)\mathcal{G}(b_\eta)+\mathcal{F}(b_\eta)\mathcal{G}(a_\eta)\\ &\preceq_{\mathrm{cr}} 2\eta(1-\eta)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\Big]\\ &\quad+\big[(1-\eta)^2+\eta^2\big]\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big].\end{aligned}$$

Substituting the preceding estimates into (3.18), we obtain

$$\begin{aligned}\mathcal{F}(m)\mathcal{G}(m) \preceq_{\mathrm{cr}} & \frac{1}{4}\Big[(1-\eta)^2+\eta^2+2\eta(1-\eta)\Big]\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)\\ & + \frac{1}{4}\Big[\eta^2+(1-\eta)^2+2\eta(1-\eta)\Big]\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\\ & + \frac{1}{4}\Big[2\eta(1-\eta)+(1-\eta)^2+\eta^2\Big]\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big]. \end{aligned} \tag{3.19}$$

Now,

$$\begin{aligned}(1-\eta)^2+\eta^2+2\eta(1-\eta)&=1,\\ \eta^2+(1-\eta)^2+2\eta(1-\eta)&=1,\end{aligned}$$

and

$$2\eta(1-\eta)+(1-\eta)^2+\eta^2=1.$$

Thus (3.19) simplifies to

$$\mathcal{F}(m)\mathcal{G}(m) \preceq_{\mathrm{cr}} \frac{1}{4}\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\frac{1}{4}\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)+\frac{1}{4}\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big]. \tag{3.20}$$

Next, multiply (3.18) first by the nonnegative kernel

$$\eta^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)$$

and integrate over $[0,1]$. This gives

$$\begin{aligned}&\left(\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\,d\eta\right)\mathcal{F}(m)\mathcal{G}(m)\\ \preceq_{\mathrm{cr}}\ & \frac{1}{4}\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{F}(b_\eta)\mathcal{G}(b_\eta)\,d\eta\\ & + \frac{1}{4}\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{F}(a_\eta)\mathcal{G}(a_\eta)\,d\eta\\ & + \frac{1}{4}\left(\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\,d\eta\right)\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\\ & + \mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big].\end{aligned}$$

Using the substitution

$$u=\zeta_2-\eta\Delta,\qquad \eta=\frac{\zeta_2-u}{\Delta},\qquad d\eta=-\frac{1}{\Delta}\,du,$$

we obtain

$$\begin{aligned}&\int_0^1 \eta^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\right)\mathcal{F}(b_\eta)\mathcal{G}(b_\eta)\,d\eta\\ &=\frac{1}{\Delta^\alpha}\int_{\zeta_1}^{\zeta_2}(\zeta_2-u)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{u}{\zeta_2}\right)\mathcal{F}(u)\mathcal{G}(u)\,du\end{aligned}$$

$$= \frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{\Delta^\alpha}\mathcal{I}_{\zeta_1^+}^{\alpha,\beta,\gamma}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2).$$

Now multiply (3.18) by the nonnegative kernel

$$\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)$$

and integrate over $[0,1]$. Using the substitution

$$v=\zeta_1+\eta\Delta,\qquad \eta=\frac{v-\zeta_1}{\Delta},\qquad d\eta=\frac{1}{\Delta}\,dv,$$

we obtain

$$\begin{aligned}
&\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\,\mathcal{F}(a_\eta)\mathcal{G}(a_\eta)\,d\eta\\
&=\frac{1}{\Delta^\alpha}\int_{\zeta_1}^{\zeta_2}(v-\zeta_1)^{\alpha-1}v^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;1-\frac{\zeta_1}{v}\bigg)\,\mathcal{F}(v)\mathcal{G}(v)\,dv\\
&=\frac{\Gamma(\alpha)}{\Delta^\alpha}\mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_1).
\end{aligned}$$

Combining these two weighted estimates and collecting the remaining endpoint terms, we arrive at

$$\begin{aligned}
&2\bigg[\int_0^1 \eta^{\alpha-1}\,{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\,d\eta\\
&+\int_0^1 \eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\,d\eta\bigg]\mathcal{F}(m)\mathcal{G}(m)\\
\preceq_{\mathrm{cr}}\ &\frac{\Gamma(\alpha)\zeta_2^{\alpha+\beta}}{2\Delta^\alpha}\mathcal{I}_{\zeta_1^+}^{\alpha,\beta,\gamma}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_2)+\frac{\Gamma(\alpha)}{2\Delta^\alpha}\mathcal{I}_{\zeta_2^-}^{\alpha,\beta,\gamma}\big(\mathcal{F}(\cdot)\mathcal{G}(\cdot)\big)(\zeta_1)\\
&+\frac{1}{2}\bigg[\int_0^1\bigg((1-\eta^2)\eta^{\alpha-1}\,{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)+(2\eta-\eta^2)\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}\\
&\times{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\bigg)\,d\eta\bigg]\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_1)\\
&+\frac{1}{2}\bigg[\int_0^1\bigg((2\eta-\eta^2)\eta^{\alpha-1}\,{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)+(1-\eta^2)\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}\\
&\times{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\bigg)\,d\eta\bigg]\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_2)\\
&+\frac{1}{2}\bigg[\int_0^1(1-\eta+\eta^2)\bigg(\eta^{\alpha-1}\,{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_2}\bigg)\\
&+\eta^{\alpha-1}(\zeta_1+\eta\Delta)^{-\alpha-\beta}{}_2F_1\bigg(\alpha+\beta,-\gamma;\alpha;\frac{\eta\Delta}{\zeta_1+\eta\Delta}\bigg)\bigg)\,d\eta\bigg]\\
&\times\Big[\mathcal{F}(\zeta_1)\mathcal{G}(\zeta_2)+\mathcal{F}(\zeta_2)\mathcal{G}(\zeta_1)\Big],
\end{aligned}$$

which is exactly (3.17). This completes the proof. □

3.2. Error bounds in numerical quadrature for interval-valued Saigo fractional operators

Moreover, convexity-based inequalities such as the Hermite–Hadamard inequality play an essential role in obtaining error estimations in numerical quadrature techniques. Indeed, the difference between the exact evaluation of the fractional integral and its midpoint- and trapezoidal-like approximations can be estimated using the properties of convexity in the underlying mapping. However, in the case of interval analysis, the error estimation is done by using inclusion relations which guarantee that the obtained results are always in accordance with the exact value of the fractional integral. Consequently, in this setting, the use of error estimations is a very useful tool in controlling the approximation errors. Such guaranteed bounds are of particular interest in computational heat transfer, where time-dependent flow and heat transport problems, such as the elliptic cylindrical flow studied in [23], are ultimately evaluated through numerical quadrature of quantities whose input data are subject to uncertainty.

Let $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}_{\mathcal{I}}^{+}$ be an interval-valued convex function of the form

$$\mathcal{F}(\zeta) = \big[\underline{\mathcal{F}}(\zeta), \overline{\mathcal{F}}(\zeta)\big],$$

where $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are convex Lebesgue integrable functions satisfying

$$0 \le \underline{\mathcal{F}}(\zeta) \le \overline{\mathcal{F}}(\zeta), \qquad \zeta \in [\zeta_1, \zeta_2].$$

Consider a partition

$$\mathcal{P} : \quad \zeta_1 = \upsilon_0 < \upsilon_1 < \cdots < \upsilon_{n-1} < \upsilon_n = \zeta_2$$

of the interval $[\zeta_1, \zeta_2]$, and for each subinterval $[\upsilon_i, \upsilon_{i+1}]$, $i = 0, 1, \ldots, n-1$, write

$$\Delta_i := \upsilon_{i+1} - \upsilon_i.$$

For parameters $\alpha > 0$ and $\beta, \gamma \in \mathcal{R}$ satisfying (H), the left-sided Saigo fractional integral on the subinterval $[\upsilon_i, \upsilon_{i+1}]$, evaluated at the right endpoint, is

$$\mathcal{I}_{\upsilon_i^+}^{\alpha,\beta,\gamma}\mathcal{F}(\upsilon_{i+1}) = \frac{\upsilon_{i+1}^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{\upsilon_i}^{\upsilon_{i+1}} (\upsilon_{i+1} - \upsilon)^{\alpha-1}\, {}_2F_1\!\left(\alpha+\beta, -\gamma; \alpha; 1 - \frac{\upsilon}{\upsilon_{i+1}}\right) \mathcal{F}(\upsilon)\, d\upsilon,$$

while the right-sided Saigo fractional integral on the same subinterval, evaluated at the left endpoint, is given by

$$\mathcal{I}_{\upsilon_{i+1}^-}^{\alpha,\beta,\gamma}\mathcal{F}(\upsilon_i) = \frac{1}{\Gamma(\alpha)} \int_{\upsilon_i}^{\upsilon_{i+1}} (\upsilon - \upsilon_i)^{\alpha-1} \upsilon^{-\alpha-\beta}\, {}_2F_1\!\left(\alpha+\beta, -\gamma; \alpha; 1 - \frac{\upsilon_i}{\upsilon}\right) \mathcal{F}(\upsilon)\, d\upsilon.$$

To construct a quadrature rule, we approximate $\mathcal{F}$ on $[\upsilon_i, \upsilon_{i+1}]$ by the linear interpolant

$$\mathcal{L}_i(\upsilon) = \mathcal{F}(\upsilon_i) + \frac{\mathcal{F}(\upsilon_{i+1}) - \mathcal{F}(\upsilon_i)}{\upsilon_{i+1} - \upsilon_i}(\upsilon - \upsilon_i).$$

Accordingly, the left- and right-sided weighted trapezoidal approximations are defined by

$$\mathcal{T}_{i,1}^{\mathrm{S}}(\mathcal{F}) = \frac{\upsilon_{i+1}^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{\upsilon_i}^{\upsilon_{i+1}} (\upsilon_{i+1} - \upsilon)^{\alpha-1}\, {}_2F_1\!\left(\alpha+\beta, -\gamma; \alpha; 1 - \frac{\upsilon}{\upsilon_{i+1}}\right) \mathcal{L}_i(\upsilon)\, d\upsilon,$$

and

$$\mathcal{T}^{\mathrm{S}}_{i,2}(\mathcal{F}) = \frac{1}{\Gamma(\alpha)} \int_{\upsilon_i}^{\upsilon_{i+1}} (\upsilon - \upsilon_i)^{\alpha-1} \upsilon^{-\alpha-\beta}\, {}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; 1 - \frac{\upsilon_i}{\upsilon}\Big)\, \mathcal{L}_i(\upsilon)\, d\upsilon.$$

The corresponding inclusion errors are defined by

$$\mathcal{E}_{i,1}(\mathcal{F}) := \mathcal{I}^{\alpha,\beta,\gamma}_{\upsilon_i^+}\mathcal{F}(\upsilon_{i+1}) - \mathcal{T}^{\mathrm{S}}_{i,1}(\mathcal{F}),$$

and

$$\mathcal{E}_{i,2}(\mathcal{F}) := \mathcal{I}^{\alpha,\beta,\gamma}_{\upsilon_{i+1}^-}\mathcal{F}(\upsilon_i) - \mathcal{T}^{\mathrm{S}}_{i,2}(\mathcal{F}).$$

For the subsequent analysis, we parameterize each subinterval affinely by

$$\upsilon = (1-\lambda)\upsilon_i + \lambda\upsilon_{i+1} = \upsilon_i + \lambda\Delta_i, \qquad \lambda \in [0,1].$$

Then

$$\upsilon_{i+1} - \upsilon = (1-\lambda)\Delta_i, \qquad \upsilon - \upsilon_i = \lambda\Delta_i.$$

Applying Theorem 3.1 on the subinterval $[\upsilon_i, \upsilon_{i+1}]$, we obtain

$$\begin{aligned}
&\left(\int_0^1 \lambda^{\alpha-1}\, {}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta_i}{\upsilon_{i+1}}\Big)\, d\lambda\right) \mathcal{F}\Big(\frac{\upsilon_i+\upsilon_{i+1}}{2}\Big) \\
&\preceq_{\mathrm{cr}} \frac{1}{2}\left[\frac{\Gamma(\alpha)\upsilon_{i+1}^{\alpha+\beta}}{\Delta_i^\alpha}\mathcal{I}^{\alpha,\beta,\gamma}_{\upsilon_i^+}\mathcal{F}(\upsilon_{i+1}) + \int_0^1 \lambda^{\alpha-1}\, {}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta_i}{\upsilon_{i+1}}\Big)\, \mathcal{F}(\upsilon_i + \lambda\Delta_i)\, d\lambda\right], \quad (3.21)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_0^1 \lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\, {}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\Big)\, d\lambda\right) \mathcal{F}\Big(\frac{\upsilon_i+\upsilon_{i+1}}{2}\Big) \\
&\preceq_{\mathrm{cr}} \frac{1}{2}\left[\int_0^1 \lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\, {}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\Big)\, \mathcal{F}(\upsilon_{i+1} - \lambda\Delta_i)\, d\lambda\right. \\
&\quad \left. + \frac{\Gamma(\alpha)}{\Delta_i^\alpha}\mathcal{I}^{\alpha,\beta,\gamma}_{\upsilon_{i+1}^-}\mathcal{F}(\upsilon_i)\right]. \quad (3.22)
\end{aligned}$$

These inclusions connect the exact Saigo fractional integrals with midpoint-type expressions on each subinterval and therefore provide a natural basis for error analysis of the corresponding weighted trapezoidal rules.

Theorem 3.5. *Let $\mathcal{F} : [\zeta_1, \zeta_2] \to \mathcal{R}^+_{\mathcal{I}}$ be an interval-valued convex mapping with*

$$\mathcal{F}(\zeta) = \big[\underline{\mathcal{F}}(\zeta), \overline{\mathcal{F}}(\zeta)\big],$$

where $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ are convex Lebesgue integrable functions, let $\alpha > 0$ and $\beta, \gamma \in \mathcal{R}$ satisfy (H)*, and let $\mathcal{P}$ be any partition of $[\zeta_1, \zeta_2]$. Then, for every subinterval $[\upsilon_i, \upsilon_{i+1}]$, the quadrature errors satisfy*

$$\begin{aligned}
\mathcal{E}_{i,1}(\mathcal{F}) \preceq_{\mathrm{cr}} &\frac{2\Delta_i^\alpha}{\Gamma(\alpha)\upsilon_{i+1}^{\alpha+\beta}} \left(\int_0^1 \lambda^{\alpha-1}\, {}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta_i}{\upsilon_{i+1}}\Big)\, d\lambda\right) \mathcal{F}\Big(\frac{\upsilon_i+\upsilon_{i+1}}{2}\Big) \\
&- \mathcal{T}^{\mathrm{S}}_{i,1}(\mathcal{F}) + \frac{\Delta_i^\alpha}{\Gamma(\alpha)\upsilon_{i+1}^{\alpha+\beta}} \int_0^1 \lambda^{\alpha-1}\, {}_2F_1\Big(\alpha+\beta, -\gamma; \alpha; \frac{\lambda\Delta_i}{\upsilon_{i+1}}\Big)\, \mathcal{F}(\upsilon_i + \lambda\Delta_i)\, d\lambda, \quad (3.23)
\end{aligned}$$

$$\begin{aligned}\mathcal{E}_{i,2}(\mathcal{F}) \preceq_{\mathrm{cr}} & \frac{2\Delta_i^{\alpha}}{\Gamma(\alpha)}\left(\int_0^1 \lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\,d\lambda\right)\mathcal{F}\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}\right)\\ & -\mathcal{T}_{i,2}^{\mathrm{S}}(\mathcal{F})\\ & +\frac{\Delta_i^{\alpha}}{\Gamma(\alpha)}\int_0^1 \lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\mathcal{F}(\upsilon_{i+1}-\lambda\Delta_i)\,d\lambda.\end{aligned} \tag{3.24}$$

Consequently, the total quadrature error

$$\mathcal{E}(\mathcal{F},\mathcal{P}) := \sum_{i=0}^{n-1}\mathcal{E}_{i,1}(\mathcal{F}) + \sum_{i=0}^{n-1}\mathcal{E}_{i,2}(\mathcal{F})$$

satisfies

$$\begin{aligned}\mathcal{E}(\mathcal{F},\mathcal{P}) \preceq_{\mathrm{cr}} \sum_{i=0}^{n-1}\Bigg\{ & \frac{2\Delta_i^{\alpha}}{\Gamma(\alpha)\upsilon_{i+1}^{\alpha+\beta}}\left(\int_0^1 \lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)\,d\lambda\right)\mathcal{F}\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}\right)\\ & -\mathcal{T}_{i,1}^{\mathrm{S}}(\mathcal{F}) + \frac{\Delta_i^{\alpha}}{\Gamma(\alpha)\upsilon_{i+1}^{\alpha+\beta}}\int_0^1 \lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)\mathcal{F}(\upsilon_i+\lambda\Delta_i)\,d\lambda\\ & +\frac{2\Delta_i^{\alpha}}{\Gamma(\alpha)}\left(\int_0^1 \lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\,d\lambda\right)\mathcal{F}\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}\right)\\ & -\mathcal{T}_{i,2}^{\mathrm{S}}(\mathcal{F})\\ & +\frac{\Delta_i^{\alpha}}{\Gamma(\alpha)}\int_0^1 \lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\mathcal{F}(\upsilon_{i+1}-\lambda\Delta_i)\,d\lambda\Bigg\}.\end{aligned} \tag{3.25}$$

Moreover, by convexity,

$$\begin{aligned} & \int_0^1 \lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)\mathcal{F}(\upsilon_i+\lambda\Delta_i)\,d\lambda\\ \preceq_{\mathrm{cr}} & \left(\int_0^1 (1-\lambda)\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)\,d\lambda\right)\mathcal{F}(\upsilon_i)\\ & +\left(\int_0^1 \lambda^{\alpha}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)\,d\lambda\right)\mathcal{F}(\upsilon_{i+1}),\end{aligned} \tag{3.26}$$

and

$$\begin{aligned} & \int_0^1 \lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\mathcal{F}(\upsilon_{i+1}-\lambda\Delta_i)\,d\lambda\\ \preceq_{\mathrm{cr}} & \left(\int_0^1 \lambda^{\alpha}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\,d\lambda\right)\mathcal{F}(\upsilon_i)\\ & +\left(\int_0^1 (1-\lambda)\lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\,d\lambda\right)\mathcal{F}(\upsilon_{i+1}).\end{aligned} \tag{3.27}$$

Substituting (3.26) *and* (3.27) *into* (3.25) *gives a purely endpoint-based inclusion estimate.*

Proof. By applying (3.21) on the subinterval $[\upsilon_i, \upsilon_{i+1}]$, we have

$$\begin{aligned}&\left(\int_0^1 \lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)\,d\lambda\right)\mathcal{F}\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}\right)\\ \preceq_{\mathrm{cr}}\ &\frac{1}{2}\left[\frac{\Gamma(\alpha)\upsilon_{i+1}^{\alpha+\beta}}{\Delta_i^{\alpha}}\mathcal{I}_{\upsilon_i^+}^{\alpha,\beta,\gamma}\mathcal{F}(\upsilon_{i+1})+\int_0^1\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)\mathcal{F}(\upsilon_i+\lambda\Delta_i)\,d\lambda\right].\end{aligned}$$

Multiplying both sides by

$$\frac{2\Delta_i^{\alpha}}{\Gamma(\alpha)\upsilon_{i+1}^{\alpha+\beta}},$$

we obtain

$$\begin{aligned}\mathcal{I}_{\upsilon_i^+}^{\alpha,\beta,\gamma}\mathcal{F}(\upsilon_{i+1})\preceq_{\mathrm{cr}}\ &\frac{2\Delta_i^{\alpha}}{\Gamma(\alpha)\upsilon_{i+1}^{\alpha+\beta}}\left(\int_0^1\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)\,d\lambda\right)\mathcal{F}\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}\right)\\ &+\frac{\Delta_i^{\alpha}}{\Gamma(\alpha)\upsilon_{i+1}^{\alpha+\beta}}\int_0^1\lambda^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)\mathcal{F}(\upsilon_i+\lambda\Delta_i)\,d\lambda.\end{aligned}$$

Subtracting $\mathcal{T}_{i,1}^{\mathrm{S}}(\mathcal{F})$ from both sides, which preserves $\preceq_{\mathrm{cr}}$ by Lemma 2.1(iii), yields (3.23).

In the same way, recalling (3.22), we have

$$\begin{aligned}&\left(\int_0^1\lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\,d\lambda\right)\mathcal{F}\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}\right)\\ \preceq_{\mathrm{cr}}\ &\frac{1}{2}\left[\int_0^1\lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\mathcal{F}(\upsilon_{i+1}-\lambda\Delta_i)\,d\lambda\right.\\ &\left.+\frac{\Gamma(\alpha)}{\Delta_i^{\alpha}}\mathcal{I}_{\upsilon_{i+1}^-}^{\alpha,\beta,\gamma}\mathcal{F}(\upsilon_i)\right].\end{aligned}$$

Multiplying both sides by $2\Delta_i^{\alpha}/\Gamma(\alpha)$, we get

$$\begin{aligned}&\mathcal{I}_{\upsilon_{i+1}^-}^{\alpha,\beta,\gamma}\mathcal{F}(\upsilon_i)\\ \preceq_{\mathrm{cr}}\ &\frac{2\Delta_i^{\alpha}}{\Gamma(\alpha)}\left(\int_0^1\lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\,d\lambda\right)\mathcal{F}\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}\right)\\ &+\frac{\Delta_i^{\alpha}}{\Gamma(\alpha)}\int_0^1\lambda^{\alpha-1}(\upsilon_i+\lambda\Delta_i)^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\gamma;\alpha;\frac{\lambda\Delta_i}{\upsilon_i+\lambda\Delta_i}\right)\mathcal{F}(\upsilon_{i+1}-\lambda\Delta_i)\,d\lambda.\end{aligned}$$

Subtracting $\mathcal{T}_{i,2}^{\mathrm{S}}(\mathcal{F})$ from both sides gives (3.24).

Summing (3.23) and (3.24) over all subintervals of the partition (Lemma 2.1(iii)) gives (3.25).

To derive the endpoint-based version, we recall that convexity yields

$$\mathcal{F}(\upsilon_i+\lambda\Delta_i)\preceq_{\mathrm{cr}}(1-\lambda)\mathcal{F}(\upsilon_i)+\lambda\mathcal{F}(\upsilon_{i+1}),$$

and

$$\mathcal{F}(\upsilon_{i+1}-\lambda\Delta_i)\preceq_{\mathrm{cr}}\lambda\mathcal{F}(\upsilon_i)+(1-\lambda)\mathcal{F}(\upsilon_{i+1}).$$

Multiplying these inequalities by the corresponding (nonnegative) Saigo kernels and integrating over $[0,1]$, we obtain (3.26) and (3.27). Substituting these bounds into (3.25), we arrive at the desired endpoint-based inclusion estimate. This completes the proof. □

Corollary 3.1 (Special case $\beta = 0$ and $\gamma = 0$)**.** *Both conditions $\beta = 0$ and $\gamma = 0$ are needed here. Indeed, for $\beta = 0$ alone one has ${}_2F_1(\alpha, -\gamma; \alpha; z) = (1-z)^{\gamma}$, so the first kernel equals $\lambda^{\alpha-1}\big(1 - \frac{\lambda\Delta_i}{\upsilon_{i+1}}\big)^{\gamma}$, which is not identically $\lambda^{\alpha-1}$ unless $\gamma = 0$. If $\beta = 0$ and $\gamma = 0$, then*

$$ {}_2F_1(\alpha, 0; \alpha; z) = 1,$$

and therefore the Saigo kernels reduce to

$$\lambda^{\alpha-1} \quad \text{and} \quad \lambda^{\alpha-1}(\upsilon_i + \lambda\Delta_i)^{-\alpha},$$

where the factor $(\upsilon_i + \lambda\Delta_i)^{-\alpha} = (\upsilon_i + \lambda\Delta_i)^{-\alpha-\beta}$ comes from $\beta = 0$. In this case the operators are the Erdélyi–Kober operators with vanishing parameter, i.e. $\mathrm{x}^{-\alpha}$ times the Riemann–Liouville integral, and (3.23), (3.24), *and* (3.25) *reduce to the corresponding weighted Riemann–Liouville type quadrature bounds. If instead $\beta = -\alpha$ and $\gamma = 0$, both kernels reduce to $\lambda^{\alpha-1}$, the factor $\upsilon_{i+1}^{\alpha+\beta}$ equals* 1*, and one recovers the (unweighted) Riemann–Liouville quadrature bounds.*

Corollary 3.2 (Special case $\alpha = 1$)**.** *If $\alpha = 1$, then the power singularity disappears, and the kernels become*

$$ {}_2F_1\left(1 + \beta, -\gamma; 1; \frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)$$

and

$$(\upsilon_i + \lambda\Delta_i)^{-1-\beta}\, {}_2F_1\left(1 + \beta, -\gamma; 1; \frac{\lambda\Delta_i}{\upsilon_i + \lambda\Delta_i}\right).$$

Hence the error estimates are governed directly by midpoint-endpoint differences weighted by the limiting hypergeometric kernels.

Corollary 3.3 (Uniform partition)**.** *If the partition is uniform, that is,*

$$\Delta_i = \upsilon_{i+1} - \upsilon_i = h = \frac{\zeta_2 - \zeta_1}{n} \qquad \text{for all } i,$$

then the total error estimate (3.25) *simplifies by replacing each Δ_i with h. In particular, the hypergeometric kernels become*

$$ {}_2F_1\left(\alpha + \beta, -\gamma; \alpha; \frac{\lambda h}{\upsilon_{i+1}}\right) \quad \text{and} \quad {}_2F_1\left(\alpha + \beta, -\gamma; \alpha; \frac{\lambda h}{\upsilon_i + \lambda h}\right).$$

Corollary 3.4 (Asymptotic behavior as $\gamma \to 0$)**.** *As $\gamma \to 0$, the hypergeometric factor tends to* 1*, namely,*

$$ {}_2F_1(\alpha + \beta, -\gamma; \alpha; z) \to 1.$$

Therefore, the Saigo quadrature bounds converge to the corresponding weighted fractional quadrature bounds without hypergeometric correction.

Corollary 3.5 (Asymptotic behavior as $\alpha \to 1$)**.** *As $\alpha \to 1$, the singularity in the power factor disappears, and the governing kernels approach*

$$ {}_2F_1\left(1 + \beta, -\gamma; 1; \frac{\lambda\Delta_i}{\upsilon_{i+1}}\right)$$

and

$$(\upsilon_i + \lambda\Delta_i)^{-1-\beta}\, {}_2F_1\left(1 + \beta, -\gamma; 1; \frac{\lambda\Delta_i}{\upsilon_i + \lambda\Delta_i}\right).$$

Hence the asymptotic quadrature error is controlled by midpoint and endpoint values weighted by these limiting hypergeometric kernels.

4. Conclusion

In this paper, we extended the Saigo fractional integral operators to the class of interval-valued functions and developed a novel framework for studying fractional integral inequalities in the interval-valued setting. Within this framework, several new Hermite–Hadamard, Fejér, and quadrature error-type inequalities were established for interval-valued convex mappings involving the Saigo operator. The obtained results emphasize the significant role of the hypergeometric kernel in linking interval-valued fractional calculus with convexity theory. Furthermore, it was shown that, under suitable choices of parameters, the derived inequalities reduce to their corresponding forms associated with classical, Riemann–Liouville, and Erdélyi–Kober fractional operators, thereby demonstrating the generality and unifying nature of the proposed approach. The applicability and effectiveness of the developed theory were illustrated through appropriate examples, remarks, and applications to numerical quadrature problems.

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