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THE ERROR ESTIMATION FOR THE LINEAR COMPLEMENTARITY PROBLEM*

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Abstract The error bounds of solutions are proposed by reformulating the linear complementarity problem (LCP) as a fixed-point equation. When the system matrix is an H_+-matrix, we estimate the values of error bounds, which make the error bounds more accurate and easy to calculate. In addition, the corresponding numerical examples are given to test the presented results.

Keywords Linear complementarity problem, P-matrix, error bound.

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1. Introduction

The linear complementarity problem arises in many application problems, such as the linear and quadratic programming problems [20, 27], the elastic contact problems in engineering area and the market equilibrium problems in economic area [6, 7, 11]. Its standard form is

$$x^{\mathrm{T}}(Ax+q)=0, x\geq 0, Ax+q\geq 0, \tag{1.1}$$

where $A=(a_{ij})\in\mathbb{R}^{n\times n}$, $q\in\mathbb{R}^n$ are given, and $x\in\mathbb{R}^n$ is to be found. Such problem is usually abbreviated as LCP(A,q) and has been studied by many researchers in recent decades, see [1, 2, 19, 32, 33] and the references therein.

Theoretical research of LCPs(A,q) includes the existence, uniqueness of solutions, and the relations with other problems, such as the variational inequality problems and the absolute value equations, see [4,7,11,18,22,23,25,28,29] for details. The LCP(A,q) with a P-matrix has a unique solution for any $q\in\mathbb{R}^n$, and such LCPs(A,q) have been explored by many researchers. Mathias and Pang presented the error bounds and the perturbation analysis for the LCP(A,q) based on a P-matrix function $c(A)$ [25]. Due to $c(A)$ is not easy to compute, many researchers looked for other ways or studied the error and perturbation problems of the LCP(A,q) with special matrices, see [3, 5, 8, 9, 13–15, 21, 24, 26, 29, 30]. Chen and Xiang proposed the new error bounds for the LCP(A,q) based on a relationship [4]. Such error bounds involved a new matrix function $\alpha_p(A)$, which was utilized by other researchers in their studies, where the bounds of $\alpha_p(A)$ were estimated, see [5, 10, 12, 14, 21] for instance. At present, in addition to the theoretical research, the research of numerical methods is also a hot topic. For the numerical solver approaches, such

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as the projection methods, the modulus-based matrix splitting type iteration methods as well as the direct methods, see [2,6,16,17,19,31,33] and the references therein.

Recently, Wu and Li proposed a modulus-based matrix splitting iteration method for solving the LCP(A,q) based on a fixed-point equation [31]. Motivated by such equivalent equation and the existing error theories, we further study the error problem of the LCP(A,q) in this paper. We first introduce some conclusions related to the P-matrix, then present the error bounds of the LCP(A,q) based on an equivalent equation. Next, for ease of calculation, we study the quantities involved in the error bounds, and present some accurate values or estimates when the system matrix A is an H_+-matrix or has special structure. Then these conclusions are applied to estimate the error bounds and some special conclusions are obtained. The main results are shown in Theorem 3.3 and Theorem 3.4 of Section 3. The numerical examples are illustrated in Sections 4 and the concluding remarks are given in Section 5.

2. Preliminaries

In this section, for the convenience of later discussion, we first briefly review some definitions, denotations, and lemmas.

Definition 2.1. ([25])A matrix $A \in \mathbb{R}^{n\times n}$ is called a P-matrix if and only if

$$\max_{1\le i\le n} x_i(Ax)_i > 0$$

for any nonzero vector $x \in \mathbb{R}^n$.

For other definitions, such as the comparison matrix $\langle A\rangle$ of $A \in \mathbb{R}^{n\times n}$, M-matrices, and H_+-matrices, see [2,21].

For the LCP(A,q), the absolute error bound given in [4] is introduced here,

$$\frac{\|r(x)\|_p}{\max_{d\in[0,1]^n}\|I-D+DA\|_p} \le \|x-x^*\|_p \le \max_{d\in[0,1]^n}\|(I-D+DA)^{-1}\|_p\|r(x)\|_p, \tag{2.1}$$

where $p \ge 1$ or $p=\infty$, x^*, $x \in \mathbb{R}^n$ denote the unique solution of LCP(A,q) and an arbitrary vector, respectively. $D = \mathrm{diag}((d_1,d_2,...,d_n))$ is a diagonal matrix with $d_i \in [0,1], i = 1,2,...,n$. $r(x)$ is the natural residual of LCP(A,q), i.e.,

$$r(x) = \min(x, Ax+q), \tag{2.2}$$

where the function "min" means to take the minimization of two vectors by componentwise.

Lemma 2.1. *Let $a = (a_1,a_2,...,a_n)^{\mathrm{T}}, b = (b_1,b_2,...,b_n)^{\mathrm{T}} \in \mathbb{R}^n$. Then there exist $d_i \in [-1,1]$ such that*

$$|a_i| - |b_i| = d_i(a_i - b_i),\ \ i = 1,2,...,n.$$

Lemma 2.2 ([10])**.** *$A \in \mathbb{R}^{n\times n}$ be an M-matrix with $a_{ii} = 1$, $i = 1,2,...,n$. Then*

$$\max_{d\in[0,1]^n}\|(I-D+DA)^{-1}\|_p = \|A^{-1}\|_p,\ \ p = 1,2,\infty, \tag{2.3}$$

where $D = diag((d_1,d_2,...,d_n))$ with $d_i \in [0,1]$, $i = 1,2,...,n$.

Lemma 2.3 ([31])**.** *Let Ω be a positive diagonal matrix. Then the LCP (A,q) is equivalent to*

$$(\Omega + A)x = |(A-\Omega)x + q| - q. \tag{2.4}$$

Lemma 2.4. *Let $A \in \mathbb{R}^{n\times n}$ be a P-matrix. Then $I - D + DA$ is a P-matrix for any $D = diag((d_1, d_2, ..., d_n))$ with $d_i \in [0, 1]$, $i = 1, 2, ..., n$.*

Lemma 2.1 and Lemma 2.4 can be easily proved, and the proof processes are omitted here. Set $\Omega = I$ in (2.4), then

$$\bar{r}(x) = (I + A)x - |(A - I)x + q| + q, \tag{2.5}$$

which is called the residual vector of the LCP (A, q), and will be utilized to estimate the error bounds of the LCP (A, q) in Section 3. It is obvious that x^* is a solution of the LCP (A, q) if and only if $\bar{r}(x^*) = 0$.

3. Main results

In this section, we discuss the error bounds through $\bar{r}(x)$ when a vector $x \in R^n$ is taken as an approximate solution of the LCP (A, q), and obtain some results.

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a P-matrix, and $x^* \in \mathbb{R}^n$ be the unique solution of the LCP(A, q). Then for any $x \in \mathbb{R}^n$,*

$$\frac{\|\bar{r}(x)\|_p}{2\max_D \|I - D + DA\|_p} \le \|x - x^*\|_p \le \frac{1}{2}\max_D \|(I - D + DA)^{-1}\|_p \|\bar{r}(x)\|_p,\ p = 1, 2, \infty, \tag{3.1}$$

where $\bar{r}(x)$ is defined by (2.5) and $D = diag((d_1, d_2, ..., d_n))$ with $d_i \in [0, 1]$, $i = 1, 2, ..., n$

Proof. Since A is a P-matrix, the LCP(A, q) has a unique solution for any $q \in R^n$. Then, for x^* and $x \in R^n$, it follows that

$$\begin{aligned}\bar{r}(x) &= (I + A)x - |(A - I)x + q| + q,\\ \bar{r}(x^*) &= (I + A)x^* - |(A - I)x^* + q| + q = 0.\end{aligned}$$

Therefore, based on Lemma 2.1, we have

$$\begin{aligned}\bar{r}(x) &= (I + A)(x - x^*) - (|(A - I)x + q| - |(A - I)x^* + q|)\\ &= (I + A)(x - x^*) - \tilde{D}((A - I)x + q - ((A - I)x^* + q))\\ &= (I + A)(x - x^*) - \tilde{D}(A - I)(x - x^*)\\ &= ((I + A) - \tilde{D}(A - I))(x - x^*)\\ &= ((I + \tilde{D}) + (I - \tilde{D})A)(x - x^*)\\ &= 2(I - D + DA)(x - x^*),\end{aligned}$$

where $\tilde{D} = \text{diag}((\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_n))$ with $\tilde{d}_i \in [-1, 1]$, $i = 1, 2, ..., n$ and $D = \text{diag}((d_1, d_2, ..., d_n))$ with $d_i \in [0, 1]$, $i = 1, 2, .., n$, $\tilde{D}$ and D satisfy $\frac{I-\tilde{D}}{2} = D$.

Since A is a P-matrix, from Lemma 2.4, we know that $I - D + DA$ is P-matrix for any $D = \text{diag}((d_1, d_2, ..., d_n))$ with $d_i \in [0, 1]$, $i = 1, 2, ..., n$. So $I - D + DA$ is a nonsingular matrix. Thus

$$x - x^* = \frac{1}{2}(I - D + DA)^{-1}\bar{r}(x).$$

So we have

$$\frac{\|\bar{r}(x)\|_p}{2\|I - D + DA\|_p} \le \|x - x^*\|_p \le \frac{1}{2}\|(I - D + DA)^{-1}\|_p \|\bar{r}(x)\|_p,\ p = 1, 2, \infty.$$

It follows that

$$\frac{\|\bar{r}(x)\|_p}{2\max_D \|I-D+DA\|_p} \le \|x-x^*\|_p \le \frac{1}{2}\max_D \|(I-D+DA)^{-1}\|_p \|\bar{r}(x)\|_p,\ p=1,2,\infty,$$

and Theorem 3.1 is established. □

Set $x=0$ in (3.1), we have the bounds of x^*, i.e., the following corollary.

Corollary 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a P-matrix, and $x^* \in \mathbb{R}^n$ be the unique solution of the LCP(A, q). Then*

$$\frac{\|q-|q|\ \|_p}{2\max_D \|I-D+DA\|_p} \le \|x^*\|_p \le \frac{1}{2}\max_D \|(I-D+DA)^{-1}\|_p \|q-|q|\ \|_p,\ p=1,2,\infty, \quad (3.2)$$

where $D = diag((d_1, d_2, ..., d_n))$ with $d_i \in [0,1]$, $i = 1, 2, ..., n$.

From Theorem 3.1 and Corollary 3.1, we can obtain the relative error bounds of $x \in \mathbb{R}^n$, that is the following Theorem 3.2.

Theorem 3.2. *Let $A \in \mathbb{R}^{n\times n}$ be a P-matrix, and $x^* \in \mathbb{R}^n$ be the unique solution of the LCP(A, q) with $x^* \neq 0$. Then for any $x \in \mathbb{R}^n$,*

$$\begin{aligned}
&\frac{\|\bar{r}(x)\|_p}{\max_D \|I-D+DA\|_p \max_D \|(I-D+DA)^{-1}\|_p \|q-|q|\ \|_p} \\
\le &\frac{\|x-x^*\|_p}{\|x^*\|_p} \\
\le &\frac{\max_D \|I-D+DA\|_p \max_D \|(I-D+DA)^{-1}\|_p \|\bar{r}(x)\|_p}{\|q-|q|\ \|_p},\ p=1,2,\infty,
\end{aligned} \quad (3.3)$$

where $\bar{r}(x)$ is defined by (2.5) and $D = diag((d_1, d_2, ..., d_n))$ with $d_i \in [0,1]$, $i = 1, 2, ..., n$.

In the following, we study $\max_D \|I-D+DA\|_p$ and $\max_D \|(I-D+DA)^{-1}\|_p$ involved in Theorem 3.1, Corollary 3.1, and Theorem 3.2 when A is some special matrix. Three conclusions are obtained, that is Lemmas 3.1-3.3.

Lemma 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be a matrix with $a_{ii} \ge 1, i = 1, 2, ..., n$. Then*

$$\begin{aligned}
&\max_D \|I-D+DA\|_1 = \|A\|_1, \\
&\max_D \|I-D+DA\|_\infty = \|A\|_\infty, \\
&\max_D \|I-D+DA\|_2 \le \|\ |A|\ \|_2, \\
&\max_D \|I-D+DA\|_2 = \|A\|_2,\ if\ A \ge 0,
\end{aligned} \quad (3.4)$$

where $D = diag((d_1, d_2, ..., d_n))$ with $d_i \in [0,1]$, $i = 1, 2, ..., n$.

Proof. Since

$$I-D+DA = I+D(A-I) = \begin{pmatrix} 1+d_1(a_{11}-1) & d_1a_{12} & \cdots & d_1a_{1n} \\ d_2a_{21} & 1+d_2(a_{22}-1) & \cdots & d_2a_{2n} \\ \vdots & \vdots & & \vdots \\ d_na_{n1} & d_na_{n2} & \cdots & 1+d_n(a_{nn}-1) \end{pmatrix},$$

it follows that

$$\begin{aligned}
\max_D \|I - D + DA\|_1 &= \max_D \max_j (1 + d_i(a_{ii} - 1) + \Sigma_{i\neq j}|d_i a_{ij}|) \\
&= \max_j \max_D (1 + d_i(a_{ii} - 1) + \Sigma_{i\neq j}|d_i a_{ij}|) \\
&= \max_j (a_{ii} + \Sigma_{i\neq j}|a_{ij}|) \\
&= \|A\|_1, \\
\max_D \|I - D + DA\|_\infty &= \max_D \max_i (1 + d_i(a_{ii} - 1) + \Sigma_{j\neq i}|d_i a_{ij}|) \\
&= \max_i \max_D (1 + d_i(a_{ii} - 1) + \Sigma_{j\neq i}|d_i a_{ij}|) \\
&= \max_i (a_{ii} + \Sigma_{j\neq i}|a_{ij}|) \\
&= \|A\|_\infty, \\
\max_D \|I - D + DA\|_2 &= \max_D \|I + D(A - I)\|_2 \\
&\leq \max_D \| \, |I + D(A - I)| \, \|_2 \\
&\leq \| \, |I + I(A - I)| \, \|_2 \\
&= \| \, |A| \, \|_2.
\end{aligned} \tag{3.5}$$

Specially, if $A \geq 0$, then $\| \, |A| \, \|_2 = \|A\|_2$. From the third expression of (3.5),

$$\max_D \|I - D + DA\|_2 \leq \|A\|_2. \tag{3.6}$$

In addition, it is obvious that

$$\max_D \|I - D + DA\|_2 \geq \|A\|_2. \tag{3.7}$$

So, from (3.6) and (3.7), when $A \geq 0$ with $a_{ii} \geq 1, i = 1, 2, ..., n$,

$$\max_D \|I - D + DA\|_2 = \|A\|_2.$$

From the above discussion, we know that the conclusion of Lemma 3.1 is established. □

Here, we give two particular cases. Set

$$A_1 = \begin{pmatrix} 1 & -0.5 & -0.4 \\ -0.1 & 1 & -0.4 \\ -0.7 & -0.4 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & -0.5 & -0.3 \\ -0.1 & 1 & -0.4 \\ -0.7 & -0.2 & 1 \end{pmatrix},$$

then A_1 and A_2 are two M-matrices with the diagonal elements being one. By calculation, we can obtain that $\max_D \|I - D + DA_1\|_2 = 1.5831 = \|A_1\|_2$ and $\max_D \|I - D + DA_2\|_2 = 1.5389$ when $D = \text{diag}((1, 0, 1))$. However, $\|A_2\|_2 = 1.5316$ and $\| \, |A_2| \, \|_2 = 1.7467$. So $\max_D \|I - D + DA_2\|_2 \neq \|A_2\|_2$. These two cases show that "$2, ...,$" in "$p = 1, 2, ..., \infty$" of Theorems 2-6 and Corollary 1 [10] should be removed to avoid minor imprecision in these places.

Lemma 3.2. *Let $A \in \mathbb{R}^{n\times n}$ be a P-matrix with $\|A - I\|_p < 1$ for $p = 1, 2, \infty$. Then*

$$\|A^{-1}\|_p \leq \max_D \|(I - D + DA)^{-1}\|_p \leq \frac{1}{1 - \|A - I\|_p}, \ p = 1, 2, \infty, \tag{3.8}$$

where $D = diag((d_1, d_2, ..., d_n))$ with $d_i \in [0, 1]$, $i = 1, 2, ..., n$.

Proof. For any $D = \mathrm{diag}((d_1, d_2, ..., d_n))$ with $d_i \in [0, 1]$, $i = 1, 2, ..., n$, the inequality

$$\|D(A - I)\|_p \leq \|A - I\|_p < 1, p = 1, 2, \infty$$

holds. It follows that

$$\|(I - D + DA)^{-1}\|_p = \|(I - D(A - I))^{-1}\|_p \leq \frac{1}{1 - \|A - I\|_p}.$$

Thus

$$|A^{-1}\|_p \leq \max_D \|(I - D + DA)^{-1}\|_p \leq \frac{1}{1 - \|A - I\|_p},\ p = 1, 2, \infty,$$

and the conclusion of Lemma 3.2 is established. □

Lemma 3.3. *Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix with $a_{ii} = 1$, $i = 1, 2, ..., n$. Then*

$$\|A^{-1}\|_p \leq \max_D \|(I - D + DA)^{-1}\|_p \leq \|\langle A\rangle^{-1}\|_p,\ p = 1, 2, \infty, \tag{3.9}$$

where $D = diag((d_1, d_2, ..., d_n))$ with $d_i \in [0, 1]$, $i = 1, 2, ..., n$.

Proof. For any $D = \mathrm{diag}((d_1, d_2, ..., d_n))$ with $d_i \in [0, 1]$, $i = 1, 2, ..., n$, we have

$$\begin{aligned}
\langle I - D + DA\rangle &= \begin{pmatrix} 1 & -d_1|a_{12}| & \cdots & -d_1|a_{1n}| \\ -d_2|a_{21}| & 1 & \cdots & -d_2|a_{2n}| \\ \vdots & \vdots & & \vdots \\ -d_n|a_{n1}| & -d_n|a_{n2}| & \cdots & 1 \end{pmatrix} \\
&\geq \begin{pmatrix} 1 & -|a_{12}| & \cdots & -|a_{1n}| \\ -|a_{21}| & 1 & \cdots & -|a_{2n}| \\ \vdots & \vdots & & \vdots \\ -|a_{n1}| & -|a_{n2}| & \cdots & 1 \end{pmatrix} \\
&= \langle A\rangle.
\end{aligned}$$

So $I - D + DA$ is an H_+-matrix. It follows that

$$0 \leq |(I - D + DA)^{-1}| \leq |\langle I - D + DA\rangle^{-1}| \leq \langle A\rangle^{-1}.$$

Thus

$$\|(I - D + DA)^{-1}\|_p \leq \|\langle A\rangle^{-1}\|_p,\ p = 1, 2, \infty.$$

On the other hand, we set $D = I$ in $\max_D \|(I - D + DA)^{-1}\|_p$, then

$$\max_D \|(I - D + DA)^{-1}\|_p \geq \|A^{-1}\|_p,\ p = 1, 2, \infty.$$

Therefore,

$$\|A^{-1}\|_p \leq \max_D \|(I - D + DA)^{-1}\|_p \leq \|\langle A\rangle^{-1}\|_p,\ p = 1, 2, \infty,$$

and the theorem is proved. □

For Lemma 3.3, there is a particular case, that is, A is an M-matrix with $a_{ii} = 1, i = 1, 2, ..., n$. Then since $A = \langle A \rangle$, the equality $A^{-1} = \langle A \rangle^{-1}$ holds. It follows that

$$\max_D \|(I - D + DA)^{-1}\|_p = \|A^{-1}\|_p, \ p = 1, 2, \infty, \tag{3.10}$$

i.e., Lemma 3.3 turns to be Lemma 2.2. Here, we supply two examples for (3.9). Set

$$A_1 = \begin{pmatrix} 1 & -0.5 & 0.3 \\ 0.5 & 1 & -0.5 \\ 0.5 & -0.5 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$$

Then A_1 is an H_+-matrix with $a_{ii} = 1, i = 1, 2, 3$, and A_2 is also an H_+-matrix but $a_{ii} \neq 1$ for $i = 1, 2, 3$. We can obtain that $\max_D \|(I - D + DA_1)^{-1}\|_2 = 2.5243$ when $D = \text{diag}((0, 1, 1))$. However, $\|A_1^{-1}\|_2 = 2.1298$ and $\|\langle A_1 \rangle^{-1}\|_2 = 14.3938$. We can find that

$$\|A_1^{-1}\|_2 \leq \max_D \|(I - D + DA_1)^{-1}\|_2 \leq \|\langle A_1 \rangle^{-1}\|_2$$

holds, but the equal sign "=" is not valid. Thus this example nicely tests the conclusion of Lemma 3.3. Through calculation, we have $\|\langle A_1 \rangle^{-1}\|_2 - \max_D \|(I - D + DA_1)^{-1}\|_2 = 11.8695$. Therefore, taking $\|\langle A \rangle^{-1}\|_2$ as the upper bound of $\max_D \|(I - D + DA)^{-1}\|_2$ sometimes results in a large error. For A_2, since it is an M-matrix too, $A_2^{-1} = \langle A_2 \rangle^{-1}$ holds. We have

$$1.5145 = \max_D \|(I - D + DA_2)^{-1}\|_2 > \|A_2^{-1}\|_2 = \|\langle A_2 \rangle^{-1}\|_2 = 1.1942.$$

Thus the inequality (3.9) in Lemma 3.3 is not satisfied. This example shows that the condition $a_{ii} = 1$ for $i = 1, 2, ..., n$ in Lemma 3.3 can not be removed.

According to Lemma 3.1 and Lemma 3.3, Theorem 3.1, Theorem 3.2, and Corollary 3.1 given before can be expressed in other forms when A is a special P-matrix, i.e., the H_+-matrix. The new results are Theorem 3.3, Corollary 3.2, and Theorem 3.4 below, which have more concise forms, do not involve the parameter matrix D, and are easy to apply. The proofs are straightforward and are omitted here.

Theorem 3.3. *Let $A \in \mathbb{R}^{n \times n}$ be an H_+-matrix with $a_{ii} = 1, i = 1, 2, ..., n$, and $x^* \in \mathbb{R}^n$ be the unique solution of the LCP(A, q). Then for any $x \in \mathbb{R}^n$,*

$$\begin{aligned} &\frac{\|\bar{r}(x)\|_p}{2\|A\|_p} \leq \|x - x^*\|_p \leq \frac{1}{2}\|\langle A \rangle^{-1}\|_p \|\bar{r}(x)\|_p, \ p = 1, \infty, \\ &\frac{\|\bar{r}(x)\|_2}{2\| \ |A| \ \|_2} \leq \|x - x^*\|_2 \leq \frac{1}{2}\|\langle A \rangle^{-1}\|_2 \|\bar{r}(x)\|_2, \end{aligned} \tag{3.11}$$

where $\bar{r}(x)$ is defined by (2.5).

Corollary 3.2. *Let $A \in \mathbb{R}^{n \times n}$ be an H_+-matrix with $a_{ii} = 1, i = 1, 2, ..., n$, and $x^* \in \mathbb{R}^n$ be the unique solution of the LCP(A, q). Then*

$$\begin{aligned} &\frac{\|q - |q| \ \|_p}{2\|A\|_p} \leq \|x^*\|_p \leq \frac{1}{2}\|\langle A \rangle^{-1}\|_p \|q - |q| \ \|_p, \ p = 1, \infty, \\ &\frac{\|q - |q| \ \|_2}{2\| \ |A| \ \|_2} \leq \|x^*\|_2 \leq \frac{1}{2}\|\langle A \rangle^{-1}\|_2 \|q - |q| \ \|_2. \end{aligned} \tag{3.12}$$

Theorem 3.4. *Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix with $a_{ii} = 1, i = 1, 2, ..., n$, and let $x^* \in \mathbb{R}^n$ be the unique solution of the LCP(A, q) with $x^* \neq 0$. Then for any $x \in \mathbb{R}^n$,*

$$\begin{aligned} &\frac{\|\bar{r}(x)\|_p}{\|A\|_p\|\langle A\rangle^{-1}\|_p\|q-|q|\ \|_p} \leq \frac{\|x-x^*\|_p}{\|x^*\|_p} \leq \frac{\|A\|_p\|\langle A\rangle^{-1}\|_p\|\bar{r}(x)\|_p}{\|q-|q|\ \|_p},\ p=1,\infty, \\ &\frac{\|\bar{r}(x)\|_2}{\|\ |A|\ \|_2\|\langle A\rangle^{-1}\|_2\|q-|q|\ \|_2} \leq \frac{\|x-x^*\|_2}{\|x^*\|_2} \leq \frac{\|\ |A|\ \|_2\|\langle A\rangle^{-1}\|_2\|\bar{r}(x)\|_2}{\|q-|q|\ \|_2}, \end{aligned} \tag{3.13}$$

where $\bar{r}(x)$ is defined by (2.5).

Remark 3.1. Comparing (3.1) with (2.1), we can find that the two expressions are very similar except that the definitions of $\bar{r}(x)$ in (2.5) and $r(x)$ in (2.2) are different. Since

$$\bar{r}(x) = (I+A)x - |(A-I)x+q| + q = x + (Ax+q) - |x-(Ax+q)| = 2\min(x, Ax+q) = 2r(x)$$

holds, so these two expressions (3.1) and (2.1) are essentially same, and we just give a different derivation process. Besides, we knew that for any positive diagonal matrix $\bar{\Omega}$, the LCP(A, q) is equivalent to

$$x^{\mathrm{T}}(\bar{\Omega}Ax + \bar{\Omega}q) = 0, x \geq 0, \bar{\Omega}Ax + \bar{\Omega}q \geq 0, \tag{3.14}$$

i.e., the LCP$(\bar{\Omega}A, \bar{\Omega}q)$. Specially, when A is an H_+-matrix, if we set $\bar{\Omega} = \Lambda^{-1}$, where Λ is the diagonal matrix of A, then the LCP(A, q) is equivalent to the LCP$(\Lambda^{-1}A, \Lambda^{-1}q)$, which satisfies the conditions of Theorem 3.3, Corollary 3.2, and Theorem 3.4. So for the LCP(A, q) with an H_+-matrix, we can estimate the error bounds of approximate solution x through the equivalent LCP$(\Lambda^{-1}A, \Lambda^{-1}q)$ with the residual vector being

$$\bar{r}_{\Lambda^{-1}}(x) = (I+\Lambda^{-1}A)x - |(\Lambda^{-1}A-I)x + \Lambda^{-1}q| + \Lambda^{-1}q = 2\min(x, \Lambda^{-1}(Ax+q)).$$

It is easy to know that we only need to replace A, q and $\bar{r}(x)$ in Theorem 3.3, Corollary 3.2, and Theorem 3.4 by $\Lambda^{-1}A$, $\Lambda^{-1}q$, and $\bar{r}_{\Lambda^{-1}}(x)$, respectively, to get some corresponding conclusions, which are omitted in this paper.

4. Numerical experiments

In this section, we show experiments to test the results given in this paper. Examples 4.1-4.2 are low order cases, and the system matrices are H_+-matrices. Examples 4.3-4.4 are high order cases, and the system matrices are M-matrices. The two high order linear complementarity problems have been studied in the literature, see [2, 32].

Example 4.1. In this example, we test the absolute error bounds (3.1) in Theorem 3.1 and the relative error bounds (3.3) in Theorem 3.2. Set A and q in the LCP(A, q) to be

$$A = \begin{pmatrix} 1 & -0.5 & -0.3 \\ 0.1 & 1 & -0.4 \\ -0.7 & -0.2 & 1 \end{pmatrix} \text{ and } q = \begin{pmatrix} -1.7 \\ 1 \\ 0.4 \end{pmatrix},$$

respectively, then A is an H_+-matrix with $a_{ii} = 1, i = 1, 2, 3$ and $x^* = (2, 0, 1)^{\mathrm{T}}$ is the unique solution of the LCP(A, q). We set

$$\bar{x} = (1.9, 0.1, 1.1)^{\mathrm{T}},\ \tilde{x} = (2, 0.1, 1.1)^{\mathrm{T}},\ \text{and } \check{x} = (2.1, 0.1, 1.2)^{\mathrm{T}}$$

Table 1. The absolute and relative error bounds.

	$\frac{\|\bar{r}(x)\|_2}{2\max_D \|I-D+DA\|_2}$	$\|x-x^*\|_2$	$\frac{1}{2}\max_D \|(I-D+DA)^{-1}\|_2\|\bar{r}(x)\|_2$
$\bar{x}$	0.1633	0.1732	0.7773
$\tilde{x}$	0.0968	0.1414	0.4607
$\check{x}$	0.0955	0.2449	0.4546
	$\frac{\|\bar{r}(x)\|_2}{\max_D \|I-D+DA\|_2 \max_D \|(I-D+DA)^{-1}\|_2\|q-\|q\|\ \|_2}$	$\frac{\|x-x^*\|_2}{\|x^*\|_2}$	$\frac{\max_D \|I-D+DA\|_2 \max_D \|(I-D+DA)^{-1}\|_2\|\bar{r}(x)\|_2}{\|q-\|q\|\ \|_2}$
$\bar{x}$	0.0315	0.0775	0.3566
$\tilde{x}$	0.0187	0.0632	0.2114
$\check{x}$	0.0184	0.1095	0.2086

to be the approximate solutions of the LCP(A, q), respectively. From Theorem 3.1 and Theorem 3.2, we obtain Table 1.

The numerical results show that Theorem 3.1 and Theorem 3.2 can be used to estimate the absolute and relative error bounds of the approximate solution x for the LCP(A, q).

Example 4.2. In this example, we test the bounds of exact solution x^*, i.e., the conclusion in Corollary 3.1. Set A and q in the LCP(A, q) to be

$$A = \begin{pmatrix} 4 & -1 \\ -2 & 2 \end{pmatrix} \text{ and } q = \begin{pmatrix} 3 \\ -2 \end{pmatrix},$$

respectively, then A is an M-matrix, and $x^* = (0, 1)^{\mathrm{T}}$ is the unique solution of the LCP(A, q). The LCP(A, q) is equivalent to the LCP$(\bar{A}, \bar{q})$ with

$$\bar{A} = \begin{pmatrix} 1 & -0.25 \\ -1 & 1 \end{pmatrix} \text{ and } \bar{q} = \begin{pmatrix} 0.75 \\ -1 \end{pmatrix}.$$

It follows that we can use the LCP$(\bar{A}, \bar{q})$ to estimate the bounds of exact solution x^*. From Corollary 3.1, we have Table 2.

Table 2. The bounds of x^*.

	$\frac{\|\bar{q}-\|\bar{q}\|\ \|_p}{2\max_D \|I-D+D\bar{A}\|_p}$	$\|x^*\|_p$	$\frac{1}{2}\max_D \|(I-D+D\bar{A})^{-1}\|_p\|\bar{q}-\|\bar{q}\|\ \|_p$
$p=1$	0.3	1	2.6667
$p=2$	0.5907	1	2.2573
$p=\infty$	0.5	1	2.6667

The numerical results in Table 2 show that Corollary 3.1 is valid in estimating the bounds of the exact solution x^*.

Example 4.3. In this example, we set A in the LCP(A,q) to be $A = \text{tridiag}(-\tau, 1+2\tau, -\tau) \in \mathbb{R}^{n\times n}$ with $\tau > 0$ and $n = 1000$, which is a tridiagonal matrix. Such LCP is derived from Black-Scholes American option pricing problem, see [32] for details. Set $q = Ae(e = (1,1,...,1)^{\mathrm{T}} \in \mathbb{R}^n)$, $\tau = 1$ and let $x = \frac{1}{s}\text{rand}(n,1)$ be the approximate solution of the LCP(A,q), where "rand" is a function in Matlab, and it can return uniformly distributed random number in the interval $(0,1)$, s is a given positive integer. In our experiments, we set $s = 1, 10, 100$, respectively. Then $x^* = (0,0,...,0)^{\mathrm{T}} \in R^n$ is the unique solution of the LCP(A,q), and the approximate solution x is closer to the exact solution x^* as the value of s increases. To estimate the error bounds of x, we use the equivalent LCP$(\bar{A},\bar{q})$, where $\bar{A} = \text{tridiag}(-\frac{\tau}{1+2\tau}, 1, -\frac{\tau}{1+2\tau})$ and $\bar{q} = \frac{\tau}{1+2\tau}q$. Since $\bar{A}$ is an M-matrix, from Lemmas 3.1 and 3.3, by calculation, we have both $\max_D \|I - D + D\bar{A}\|_p = \|\bar{A}\|_p$ and $\max_D \|(I - D + D\bar{A})^{-1}\|_p = \|\bar{A}^{-1}\|_p$ for $p = 1, 2, \infty$. According to Theorems 3.1 and 3.3, the numerical results are shown in Table 3.

Table 3. The absolute error bounds of x.

	$\frac{\|\bar{r}_{\Lambda^{-1}}(x)\|_p}{2\|\bar{A}\|_p}$	$\|x - x^*\|_p$	$\frac{1}{2}\|\bar{A}^{-1}\|_p\|\bar{r}_{\Lambda^{-1}}(x)\|_p$	$\|\bar{r}_{\Lambda^{-1}}(x)\|_p$
$p = 1$	0.2735*1.0e+3	0.5037*1.0e+3	1.3675*1.0e+3	0.9117*1.0e+3
$p = 2$	10.1969	18.4010	50.9841	33.9898
$p = \infty$	0.5981	0.9984	2.9906	1.9938
$p = 1$	29.3300	48.8833	146.6498	97.7665
$p = 2$	1.0718	1.7863	5.3590	3.5727
$p = \infty$	0.0600	0.0999	0.2998	0.19992
$p = 1$	3.0601	5.1002	15.3006	10.2004
$p = 2$	0.1109	0.1848	0.5543	0.3696
$p = \infty$	0.0060	0.0100	0.0300	0.0200

We ran the function "rand" three times in our experiments, that is, $s = 1, 10, 100$, respectively. From Table 3, we can observe that the numerical results are similar, and Theorems 3.1 and 3.3 can be applied to estimate the absolute error bounds of the approximate solution.

Example 4.4. In this example, we consider high order cases for (3.11) in Theorem 3.3 and (3.13) in Theorem 3.4. Set A to be $A = \text{Tridiag}(-I, S, -I) \in \mathbb{R}^{n\times n}$, which is a block tridiagonal M-matrix with $S = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{m\times m}$ and I being a tridiagonal matrix and an identity matrix, respectively, positive integers m and n satisfy $n = m^2$. For convenience, we set the exact solution of the LCP(A,q) to be $x^{(*)} = (0,1,0,1,...)^{\mathrm{T}} \in R^n$ and $q = -Ax^{(*)}$. The numerical solutions of the LCP(A,q) are generated by the modulus-based Gauss-Seidel (MGS) iteration method, which is an effective iterative method for solving the LCP(A,q). For such LCP and the MGS iteration method, see [2] for details. Let the parameter matrix Ω be the diagonal part of A

in the MGS iteration method, and let the initial iteration vector be $x^{(0)} = (1, 0, 1, 0, ...)^{\mathrm{T}} \in \mathbb{R}^n$ and set the iteration process to cease when $\|r(x^{(k)})\|_2 < 10^{-5}$, where $r(x^{(k)}) = \min(x^{(k)}, Ax^{(k)} + q)$ and $x^{(k)}$ is the kth numerical solution. Set $n = 900$, and consider $\|x^{(k)} - x^{(*)}\|_p$ and $\frac{\|x^{(k)} - x^{(*)}\|_p}{\|x^{(*)}\|_p}$ for $p = 1, 2, \infty$. Based on Theorems 3.3-3.4, the numerical results are shown in Figure 1 and Figure 2.

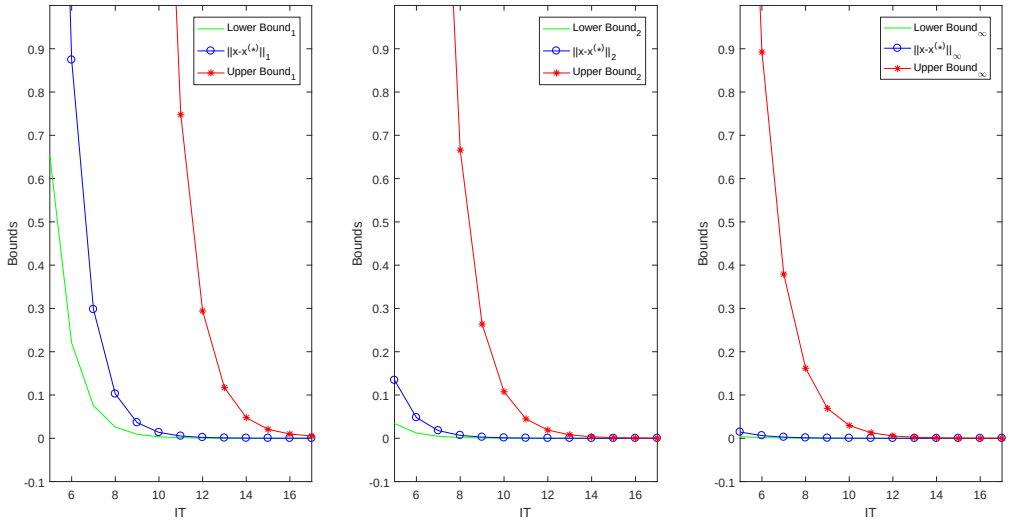


Figure 1. The absolute error bounds of the numerical solution $x^{(k)}$.

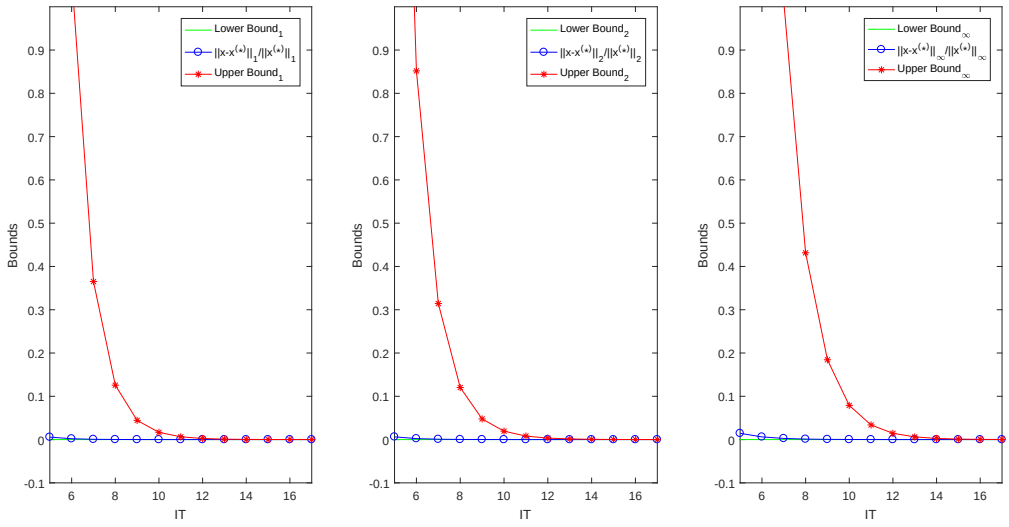


Figure 2. The relative error bounds of the numerical solution $x^{(k)}$.

The horizontal axis presents the number of iterative steps in Figures 1-2. The "Lower Bound" line and "Upper Bound" line in Figure 1 represent functions

$$\frac{\|\bar{r}_{\Lambda^{-1}}(x^{(k)})\|_p}{2\|\Lambda^{-1}A\|_p} \text{ and } \frac{1}{2}\|(\Lambda^{-1}A)^{-1}\|_p\|\bar{r}_{\Lambda^{-1}}(x^{(k)})\|_p$$

for $p = 1, 2, \infty$, respectively, and the "Lower Bound" line and "Upper Bound" line in Figure 2 represent functions

$$\frac{\|\bar{r}_{\Lambda^{-1}}(x^{(k)})\|_p}{\|\Lambda^{-1}A\|_p\|(\Lambda^{-1}A)^{-1}\|_p\|\Lambda^{-1}q - |\Lambda^{-1}q|\ \|_p} \text{ and } \frac{\|\Lambda^{-1}A\|_p\|(\Lambda^{-1}A)^{-1}\|_p\|\bar{r}_{\Lambda^{-1}}(x^{(k)})\|_p}{\|\Lambda^{-1}q - |\Lambda^{-1}q|\ \|_p}$$

for $p = 1, 2, \infty$, respectively, where Λ is the diagonal matrix of A so that $\Lambda^{-1}A$ is an M-matrix with its diagonal matrix being I, and Theorems 3.3-3.4 can be applied to the equivalent LCP$(\Lambda^{-1}A, \Lambda^{-1}q)$ to estimate the error bounds of LCP(A, q). We note that for the linear complementarity problem LCP$(\Lambda^{-1}A, \Lambda^{-1}q)$, $\max_D \|I - D + D(\Lambda^{-1}A)\|_p = \|\Lambda^{-1}A\|_p = \|\ |\Lambda^{-1}A|\ \|_p$ and $\max_D \|(I - D + D(\Lambda^{-1}A))^{-1}\|_p = \|(\Lambda^{-1}A)^{-1}\|_p$ for $p = 1, 2, \infty$ in this example.

5. Concluding remarks

By utilizing the equivalent equation of the LCP(A, q), we discuss the absolute and relative error problems of the approximate solution. When the system matrix A is a P-matrix and its special case, that is the H_+-matrix, some conclusions are given. The numerical experiments are illustrated to verify the error bound results.

Conflict of interest. The authors have no conflict of interest.

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