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POSITIVE SOLUTIONS OF A HADAMARD-TYPE FRACTIONAL-ORDER DIFFERENTIAL SYSTEM WITH HADAMARD INTEGRAL AND DISCRETE BOUNDARY CONDITIONS ON THE HALF-LINE*

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Abstract In the present work, we investigate a class of Hadamard-type fractional differential system subject to both Hadamard integral and discrete boundary conditions on the half-line. By defining a new partial order on the product space and coupling with the application of monotone iterative methods, we construct two distinct pairs of explicit monotone iterative sequences that exhibit convergence toward the minimal and maximal positive solutions. A concrete example is provided to demonstrate the application of the theorem.

Keywords Monotone iterative method, fractional-order differential system, nonlocal conditions, unbounded interval.

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1. Introduction

In the present study, we focus on analyzing a Hadamard-type fractional-order differential system (HFDS) defined as follows:

$$\begin{cases} {}^{H}D^{\delta_1}u(x)+e_1(x,u(x),v(x),{}^{H}D^{\delta_1-1}u(x),{}^{H}D^{\delta_2-1}v(x))=0, n_1-1<\delta_1\le n_1,\\ {}^{H}D^{\delta_2}v(x)+e_2(x,u(x),v(x),{}^{H}D^{\delta_1-1}u(x),{}^{H}D^{\delta_2-1}v(x))=0, n_2-1<\delta_2\le n_2,\\ u^{(h)}(1)=0(h=0,1,2,\cdots,n_1-2),\ v^{(k)}(1)=0(k=0,1,2,\cdots,n_2-2),\\ {}^{H}D^{\delta_1-1}u(\infty)=\displaystyle\sum_{i=1}^{q_1}\nu_i{}^{H}I^{\beta_i}u(\varrho_1)+b_1\sum_{j=1}^{p_1}\lambda_j u(\zeta_j),\\ {}^{H}D^{\delta_2-1}v(\infty)=\displaystyle\sum_{\varsigma=1}^{q_2}\mu_\varsigma{}^{H}I^{\alpha_\varsigma}v(\varrho_2)+b_2\sum_{\tau=1}^{p_2}\pi_\tau v(\eta_\tau), \end{cases} \tag{1.1}$$

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where D^{ϕ} denote the fractional-order derivatives of Hadamard type for $\phi \in \{\delta_1, \delta_2, \delta_1 - 1, \delta_2 - 1\}, e_1, e_2 \in C(J \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+), J = (1, +\infty), \mathbb{R}_+ = [0, +\infty)$, I^{ψ} denote the fractional-order derivatives of Hadamard type for $\psi \in \{\beta_i, \alpha_\varsigma\}, \beta_i, \alpha_\varsigma > 0, 1 < \varrho_1 < \zeta_1 < \zeta_2 < \cdots < \zeta_{p_1} < +\infty, 1 < \varrho_2 < \eta_1 < \eta_2 < \cdots < \eta_{p_2} < +\infty, \nu_i, \mu_\varsigma, \lambda_j, \pi_\tau, b_1, b_2 \geq 0 (i = 1, 2, \ldots, q_1; \varsigma = 1, 2, \ldots, q_2; j = 1, 2, \ldots, p_1; \tau = 1, 2, \ldots, p_2)$ are given constants with

$$\Gamma(\delta_1) - \sum_{i=1}^{q_1} \nu_i \frac{\Gamma(\delta_1)}{\Gamma(\delta_1 + \beta_i)} (\ln \varrho_1)^{\delta_1 + \beta_i - 1} - b_1 \sum_{j=1}^{p_1} \lambda_j (\ln \zeta_j)^{q_1 - 1} = \Lambda_1 > 0,$$

and

$$\Gamma(\delta_2) - \sum_{\varsigma=1}^{q_2} \mu_\varsigma \frac{\Gamma(\delta_2)}{\Gamma(\delta_2 + \alpha_\varsigma)} (\ln \varrho_2)^{\delta_2 + \alpha_\varsigma - 1} - b_2 \sum_{\tau=1}^{p_2} \pi_\tau (\ln \eta_\tau)^{q_2 - 1} = \Lambda_2 > 0.$$

The main purpose of this paper is to meticulously construct two distinct sets of explicit monotone iterative sequences. These sequences are designed in such a way that they will converge respectively to the minimal and the maximal positive solutions.

Fractional-order calculus, which is as old as the integer-order one, has attracted much attention in recent decades. The reason is that fractional-order differential equations have been proved to be an indispensable and powerful tool for accurately modeling complex real-world phenomena. Fractional-order differential equations are used in a wide range of fields such as continuum mechanics, thermodynamic physics, quantum mechanical systems, anomalous diffusion processes, control theory [6,7,11,12,20], etc. As a generalization of classical integer-order differential equations, fractional-order differential equations can provide more accurate descriptions and predictions for theoretical and practical problems. There are many definitions of fractional-order operators with different properties for various applications, the Riemann-Liouville fractional derivative, the Caputo fractional derivative, the Hadamard fractional derivative and other common fractional derivative definitions. This paper selects a specific type of differential system based on the so-called Hadamard-type fractional operator, which was first proposed by Hadamard in [5]. It has been widely investigated and applied in digital image processing, materials fracture analysis [1,2,4,8,21,23], and references therein.

Numerous results exist concerning Hadamard-type fractional-order differential equation (HFDE), the majority of which have been established using various fixed point theorems [9, 10, 14]. For instance, in the research work cited as [3], Ahmad and Ntouyas delved into the following HFDS on a bounded interval. They skillfully employed the Banach contraction mapping principle along with the Leray-Schauder fixed point theorem. Through these rigorous mathematical tools, they managed to obtain both the existence and the uniqueness of solutions:

$$\begin{cases} {}^{H}D^{\alpha} u(s) = f(x, u(x), v(x)), \ 1 < \alpha \leq 2, \ 1 < x < e, \\ {}^{H}D^{\beta} v(x) = g(x, u(x), v(x)), \ 1 < \beta \leq 2, \ 1 < x < e, \\ u(1) = 0, \ u(e) = {}^{H}I^{r} u(\sigma_1), \\ v(1) = 0, \ v(e) = {}^{H}I^{r} v(\sigma_2). \end{cases} \tag{1.2}$$

By utilizing the Leggett-Williams fixed point theorem, Xu et al. in [19] considered the following HFDS:

$$\begin{cases} {}^{H}D^{p}u(x) + f_1(x, u(x), v(x)) = 0,\ 2 < q \le 3,\ 1 < x \le e, \\ {}^{H}D^{q}v(x) + f_2(x, u(x), v(x)) = 0,\ 2 < q \le 3,\ 1 < x \le e, \\ u(1) = \delta u(1) = 0,\ u(e) = \sum_{i=1}^{m-1} a_i u(\xi_i), \\ v(1) = \delta v(1) = 0,\ v(e) = \sum_{j=1}^{n-1} b_j v(\eta_j). \end{cases} \tag{1.3}$$

The existence of three solutions was confirmed.

Wang et al. [17] made use of the Leggett-Williams fixed point theorem to conduct an in-depth analysis of the following HFDE:

$$\begin{cases} {}^{H}D^{q}u(x) + a(x)f(x, u(x)) = 0, 2 < \alpha < 3, x \in (1, +\infty), \\ u(1) = u'(1) = 0, {}^{H}D^{\alpha-1}u(\infty) = \alpha\, {}^{H}I^{\beta}u(\eta) + b\sum_{j=1}^{n} \sigma_j u(\xi_j), \end{cases} \tag{1.4}$$

and rigorously established the sufficient conditions that guarantee the existence of solutions for this system.

Based on the integrated boundary conditions developed in [16], the research team behind [22] examined the solvability characteristics of positive solutions associated with a more comprehensive generalization of the HFDE under extended boundary condition frameworks:

$$\begin{cases} {}^{H}D^{q}u(x) + a(x)f(x, u(x)) = 0, 2 < \alpha < 3, x \in (1, +\infty), \\ u(1) = u'(1) = 0, {}^{H}D^{\alpha-1}u(\infty) = \sum_{i=1}^{m} \alpha_i\, {}^{H}I^{\beta_i}u(\eta) + b\sum_{j=1}^{n} \sigma_j u(\xi_j). \end{cases} \tag{1.5}$$

Motivated by the aforementioned significant results, this paper investigates the HFDS (1.1). As far as we are aware, there have been relatively few studies that have investigated the properties and solution strategies of the system (1.1) on the half-line. This research gap constitutes the primary motivation for the present work. In comparison to existing literature, this paper offers several key contributions: Firstly, the boundary conditions of the HFDS (1.1) involve more general Hadamard integral boundary conditions and discrete boundary conditions, which not only extend the scope of the research but also generalize the corresponding conclusions in [16,18, 22]. Secondly, a novel extension of the classical iterative technique is applied to solve the HFDS (1.1), wherein the method is implemented in a product space under a new partial order, differing from conventional approaches for single equations (see [15,17,22]). Thirdly, novel characteristics of the Green's function denoted as $\kappa(x, s, \delta)$ are established (see Lemma 2.4, properties (B1) and (B2)). Based on these properties, a series of important theoretical results concerning the HFDS (1.1) are derived (see Lemma 3.2 and Theorem 3.1).

This research article is structured in the following manner: Chapter 2 introduces some necessary definitions and fundamental lemmas for deriving the main results. Chapter 3 presents our main results, in which two explicit positive solution sets of the HFDS (1.1) are constructed and analyzed. A numerical example is also given to demonstrate the obtained results.

2. Preliminaries

To facilitate the subsequent proof process, we initially present essential definitions and summarize relevant preliminary results that serve as the theoretical basis for establishing our main findings.

Definition 2.1. (See [2, 6]) The Hadamard-type fractional derivative of order $q > 0$ for an integrable function $g : J \to \mathbb{R}$ is defined as

$$^{H}D^{q}f(x) = \frac{1}{\Gamma(n-q)}\left(x\frac{d}{dt}\right)^{n}\int_{1}^{x}(\ln x - \ln s)^{n-q-1}\,f(s)\frac{\mathrm{d}s}{s},\ x > 1,$$

where $n = [q] + 1$, $[q]$ represents the integer part of the real number q.

Definition 2.2. (See [2, 6]) The Hadamard-type fractional integral of order $q > 0$ for an integrable function $g : J \to \mathbb{R}$ is defined as

$$^{H}I^{q}f(x) = \frac{1}{\Gamma(q)}\int_{1}^{x}(\ln x - \ln s)^{q-1}\,f(s)\frac{\mathrm{d}s}{s},\ x > 1.$$

Let us define two spaces consisting of continuous functions in the following manner:

$$U = \left\{u \in C(J), {}^{H}D^{\delta_1-1}u \in C(J) : \sup_{x\in J}\frac{|u(x)|}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}} < +\infty,\ \sup_{x\in J}|{}^{H}D^{\delta_1-1}u(x)| < +\infty\right\},$$

$$V = \left\{v \in C(J), {}^{H}D^{\delta_2-1}v \in C(J) : \sup_{x\in J}\frac{|v(x)|}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}} < +\infty,\ \sup_{x\in J}|{}^{H}D^{\delta_2-1}v(x)| < +\infty\right\}$$

fully equipped with well-defined norms

$$\|u\|_U = \max\left\{\sup_{s\in J}\frac{|u(x)|}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}, \sup_{x\in J}|{}^{H}D^{\delta_1-1}u(x)|\right\},$$

$$\|v\|_V = \max\left\{\sup_{x\in J}\frac{|v(x)|}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}, \sup_{x\in J}|{}^{H}D^{\delta_2-1}v(x)|\right\}.$$

As indicated in the reference [13], $(U, ||\cdot||_U)$ and $(V, ||\cdot||_V)$ are two Banach spaces. Based on this fact, the conclusion of Lemma 2.1 can be readily derived.

Lemma 2.1. *The product space $(U \times V, \|\cdot\|_{U\times V})$ furnished with the norm $\|\cdot\|_{U\times V} = \max\{\|u\|_U, \|v\|_V\}$ stands a Banach space.*

Lemma 2.2. *If the continuous functions e_1 and e_2 satisfy the condition that $0 < \int_1^\infty e_i(s)\frac{\mathrm{d}s}{s} < \infty$ for $i = 1, 2$, and given that Λ_1 and Λ_2 are positive real numbers, then a function pair $(u(t), v(t))$ belonging to the product space $U \times V$ is a solution of the differential system*

$$\begin{cases} {}^{H}D^{\delta_1}u(x) + e_1(x) = 0, n_1 - 1 < \delta_1 \le n_1, x \in (1, +\infty), \\ {}^{H}D^{\delta_2}v(x) + e_2(x) = 0, n_2 - 1 < \delta_2 \le n_2, x \in (1, +\infty), \\ u^{(h)}(1) = 0 (h = 0, 1, 2, \cdots, n_1 - 2), v^{(k)}(1) = 0 (k = 0, 1, 2, \cdots, n_2 - 2), \\ {}^{H}D^{\delta_1-1}u(\infty) = \sum_{i=1}^{q_1}\nu_i\,{}^{H}I^{\beta_i}u(\varrho_1) + b_1\sum_{j=1}^{p_1}\lambda_j u(\zeta_j), \\ {}^{H}D^{\delta_2-1}v(\infty) = \sum_{\varsigma=1}^{q_2}\mu_\varsigma\,{}^{H}I^{\alpha_\varsigma}v(\varrho_2) + b_2\sum_{\tau=1}^{p_2}\pi_\tau v(\eta_\tau), \end{cases} \tag{2.1}$$

if and only if $(u(x), v(x))$ *satisfies the following integral system:*

$$\begin{cases} u(x) = \displaystyle\int_1^{+\infty} K_1(x,s)e_1(s)\frac{\mathrm{d}s}{s}, \ x \in J, \\ v(x) = \displaystyle\int_1^{+\infty} K_2(x,s)e_2(s)\frac{\mathrm{d}s}{s}, \ x \in J, \end{cases} \tag{2.2}$$

where $K_i(t,s), i = 1,2$ *are specified as follows:*

$$K_1(x,s) = \kappa(x,s,\delta_1) + \frac{(\ln x)^{\delta_1-1}}{\Lambda_1}\sum_{i=1}^{q_1}\nu_i\kappa(\varrho_1,s,\delta_1+\beta_i) + \frac{(\ln t)^{\delta_1-1}}{\Lambda_1}\sum_{j=1}^{p_1}b_1\lambda_j\kappa(\zeta_j,s,\delta_1), \ x \in J,$$

$$K_2(x,s) = \kappa(x,s,\delta_2) + \frac{(\ln t)^{\delta_2-1}}{\Lambda_2}\sum_{\varsigma=1}^{q_2}\mu_\varsigma\kappa(\varrho_2,s,\delta_2+\alpha_\varsigma) + \frac{(\ln x)^{\delta_2-1}}{\Lambda_2}\sum_{\tau=1}^{p_2}b_2\pi_\tau\kappa(\eta_\tau,s,\delta_2), \ x \in J,$$

with

$$\kappa(x,s,\delta) = \frac{1}{\Gamma(\delta)}\begin{cases} (\ln x)^{\delta-1} - (\ln x - \ln s)^{\delta-1}, \ 1 \le s \le x < +\infty, \\ (\ln x)^{\delta-1}, \ 1 \le x \le s < +\infty. \end{cases} \tag{2.3}$$

Proof. Through the application of Lemma 2.2 and Lemma 3.1 presented in reference [22], the results mentioned earlier can be straightforwardly derived. □

Lemma 2.3. *If the continuous functions* e_1 *and* e_2 *satisfy the condition that* $0 < \int_1^{\infty} e_i(s)\frac{\mathrm{d}s}{s} < \infty$ *for* $i = 1,2$*, and given that* Λ_1 *and* Λ_2 *are positive real numbers, then the following results may be derived from* (2.2)

$$\begin{cases} {}^H D^{\delta_1-1}u(x) = \displaystyle\int_1^{+\infty} K_1^*(x,s)e_1(s)\frac{\mathrm{d}s}{s}, \\ {}^H D^{\delta_2-1}v(x) = \displaystyle\int_1^{+\infty} K_2^*(x,s)e_2(s)\frac{\mathrm{d}s}{s}, \end{cases} \tag{2.4}$$

where $K_i^*(x,s), i = 1,2$ *are specified as follows:*

$$K_1^*(x,s) = K_0(x,s) + \frac{\Gamma(\delta_1)}{\Lambda_1}\sum_{i=1}^{q_1}\nu_i\kappa(\varrho_1,s,\delta_1+\beta_i) + \frac{\Gamma(\delta_1)}{\Lambda_1}\sum_{j=1}^{p_1}b_1\lambda_j\kappa(\zeta_j,s,\delta_1),$$

$$K_2^*(x,s) = K_0(x,s) + \frac{\Gamma(\delta_2)}{\Lambda_2}\sum_{\varsigma=1}^{q_2}\mu_\varsigma\kappa(\varrho_2,s,\delta_2+\alpha_\varsigma) + \frac{\Gamma(\delta_2)}{\Lambda_2}\sum_{\tau=1}^{p_2}b_2\pi_\tau\kappa(\eta_\tau,s,\delta_2),$$

and

$$K_0(x,s) = \begin{cases} 0, 1 \le s \le x < +\infty, \\ 1, 1 \le x \le s < +\infty. \end{cases} \tag{2.5}$$

Proof. Differentiating with respect to (2.2), it can be directly calculated that

$${}^H D^{\delta_1-1}u(x) = c_1\Gamma(\delta_1) - \int_1^x e_1(s)\frac{\mathrm{d}s}{s},$$

$${}^H D^{\delta_2-1}v(x) = k_1\Gamma(\delta_2) - \int_1^x e_2(s)\frac{\mathrm{d}s}{s},$$

where

$$c_1 = \frac{1}{\Gamma(\delta_1)} \int_1^{\infty} e_1(s) \frac{\mathrm{d}s}{s} + \frac{1}{\Lambda_1} \sum_{i=1}^{q_1} \nu_i \int_1^{+\infty} \kappa(\varrho_1, s, \delta_1 + \beta_i) e_1(s) \frac{\mathrm{d}s}{s} + \frac{1}{\Lambda_1} \sum_{j=1}^{p_1} b_1 \lambda_j \int_1^{+\infty} \kappa(\zeta_j, s, \delta_1) e_1(s) \frac{\mathrm{d}s}{s},$$

and

$$k_1 = \frac{1}{\Gamma(\delta_2)} \int_1^{\infty} e_2(s) \frac{\mathrm{d}s}{s} + \frac{1}{\Lambda_2} \sum_{\varsigma=1}^{q_2} \mu_\varsigma \int_1^{+\infty} \kappa(\varrho_2, s, \delta_2 + \alpha_\varsigma) e_2(s) \frac{\mathrm{d}s}{s} + \frac{1}{\Lambda_2} \sum_{\tau=1}^{p_2} b_2 \pi_\tau \int_1^{+\infty} \kappa(\eta_\tau, s, \delta_2) e_2(s) \frac{\mathrm{d}s}{s}.$$

Substituting c_1 and k_1 into the expression ${}^H D^{\delta_1 - 1} u(t)$ and ${}^H D^{\delta_2 - 1} v(t)$ yields

$$\begin{aligned} {}^H D^{\delta_1 - 1} u(x) &= \int_x^{\infty} e_1(s) \frac{\mathrm{d}s}{s} + \frac{\Gamma(\delta_1)}{\Lambda_1} \sum_{i=1}^{q_1} \nu_i \int_1^{+\infty} \kappa(\varrho_1, s, \delta_1 + \beta_i) e_1(s) \frac{\mathrm{d}s}{s} \\ &\quad + \frac{\Gamma(\delta_1)}{\Lambda_1} \sum_{j=1}^{p_1} b_1 \lambda_j \int_1^{+\infty} \kappa(\zeta_j, s, \delta_1) e_1(s) \frac{\mathrm{d}s}{s} \\ &= \int_1^{\infty} K_0(x, s) e_1(s) \frac{\mathrm{d}s}{s} + \frac{\Gamma(\delta_1)}{\Lambda_1} \sum_{i=1}^{q_1} \nu_i \int_1^{+\infty} \kappa(\varrho_1, s, \delta_1 + \beta_i) e_1(s) \frac{\mathrm{d}s}{s} \\ &\quad + \frac{\Gamma(\delta_1)}{\Lambda_1} \sum_{j=1}^{p_1} b_1 \lambda_j \int_1^{+\infty} \kappa(\zeta_j, s, \delta_1) e_1(s) \frac{\mathrm{d}s}{s} \\ &= \int_1^{\infty} K_1^*(x, s) e_1(s) \frac{\mathrm{d}s}{s}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} {}^H D^{\delta_2 - 1} v(x) &= \int_1^{\infty} K_0(x, s) e_2(s) \frac{\mathrm{d}s}{s} + \frac{\Gamma(\delta_2)}{\Lambda_2} \sum_{\varsigma=1}^{q_2} \mu_\varsigma \int_1^{+\infty} \kappa(\varrho_2, s, \delta_2 + \alpha_\varsigma) e_2(s) \frac{\mathrm{d}s}{s} \\ &\quad + \frac{\Gamma(\delta_2)}{\Lambda_2} \sum_{\tau=1}^{p_2} b_2 \pi_\tau \int_1^{+\infty} \kappa(\eta_\tau, s, \delta_2) e_2(s) \frac{\mathrm{d}s}{s} \\ &= \int_1^{\infty} K_2^*(x, s) e_2(s) \frac{\mathrm{d}s}{s}. \end{aligned} \tag{2.7}$$

This shows that the expressions (2.6) and (2.7) are consistent with (2.4), thus the proof is complete. □

For brevity, we introduce the following notations:

$$M_1 = \frac{1}{\Gamma(\delta_1)} + \frac{1}{\Lambda_1} \sum_{i=1}^{q_1} \nu_i (\ln \varrho_1)^{\delta_1 + \beta_i} + \frac{1}{\Lambda_1} \sum_{j=1}^{p_1} b_1 \lambda_j (\ln \zeta_j)^{\delta_1},$$

$$M_2 = \frac{1}{\Gamma(\delta_2)} + \frac{1}{\Lambda_2} \sum_{\varsigma=1}^{q_2} \mu_\varsigma (\ln \varrho_2)^{\delta_2 + \alpha_\varsigma} + \frac{1}{\Lambda_2} \sum_{\tau=1}^{p_2} b_2 \pi_\tau (\ln \eta_\tau)^{\delta_2},$$

$$N_1 = 1 + \frac{\Gamma(\delta_1)}{\Lambda_1}\sum_{i=1}^{q_1}\nu_i(\ln\varrho_1)^{\delta_1+\beta_i} + \frac{\Gamma(\delta_1)}{\Lambda_1}\sum_{j=1}^{p_1}b_1\lambda_j(\ln\zeta_j)^{\delta_1} = \Gamma(\delta_1)M_1,$$

$$N_2 = 1 + \frac{\Gamma(\delta_2)}{\Lambda_2}\sum_{\varsigma=1}^{q_2}\mu_\varsigma(\ln\varrho_2)^{\delta_2+\alpha_\varsigma} + \frac{\Gamma(\delta_2)}{\Lambda_2}\sum_{\tau=1}^{p_2}b_2\pi_\tau(\ln\eta_\tau)^{\delta_2} = \Gamma(\delta_2)M_2.$$

Lemma 2.4. *The kernel functions $K_i(x,s)$ for $i=1,2$ defined in equation (2.2) possess the following analytical characteristics:*

(C1) *For each $i=1,2, K_i(x,s)$ represents a continuous non-negative function over the domain $J\times J$.*

(C2) *There exist positive constants $M_i, i=1,2$ such that the inequality*

$$\frac{K_i(x,s)}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}} \le M_i, \quad (x,s)\in J\times J$$

holds.

Proof. Property (C1) is clearly defined and can be obtained directly through calculation.

Notably,

$$\begin{aligned}
&\frac{K_1(x,s)}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}\\
\le&\Big[\frac{1}{\Gamma(\delta_1)}+\frac{1}{\Lambda_1}\sum_{i=1}^{q_1}\nu_i\kappa(\varrho_1,s,\delta_1+\beta_i)+\frac{1}{\Lambda_1}\sum_{j=1}^{p_1}b_1\lambda_j\kappa(\zeta_j,s,\delta_1)\Big]\frac{(\ln x)^{\delta_1-1}}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}\\
\le&\frac{1}{\Gamma(\delta_1)}+\frac{1}{\Lambda_1}\sum_{i=1}^{q_1}\nu_i\kappa(\varrho_1,\varrho_1,\delta_1+\beta_i)+\frac{1}{\Lambda_1}\sum_{j=1}^{p_1}b_1\lambda_j\kappa(\zeta_j,\zeta_j,\delta_1)\\
=&M_1,(x,s)\in J\times J,\\
&\frac{K_2(x,s)}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}\\
\le&\Big[\frac{1}{\Gamma(\delta_2)}+\frac{1}{\Lambda_2}\sum_{\varsigma=1}^{p_1}\mu_\varsigma\kappa(\varrho_2,\varrho_2,\delta_2+\alpha_\varsigma)+\frac{1}{\Lambda_2}\sum_{\tau=1}^{p_2}b_2\pi_\tau\kappa(\eta_\tau,\eta_\tau,\delta_2)\Big]\frac{(\ln x)^{\delta_2-1}}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}\\
\le&M_2,(x,s)\in J\times J,
\end{aligned}$$

thus, property (C2) is satisfied. □

Lemma 2.5. *The integral kernels $K_i(x,s)$ and $K_i^*(x,s)$ for $i=1,2$, which are introduced in equations (2.2) and (2.4), possess the subsequent mathematical characteristics:*

(D1) *$K_1(x,s)\le M_1(\ln x)^{\delta_1-1}; K_2(x,s)\le M_2(\ln x)^{\delta_2-1}$ for $(x,s)\in J\times J$.*

(D2) *$0\le K_i^*(x,s)\le N_i$ for $(x,s)\in J\times J$.*

Proof. From the expressions given in (2.3) and (2.5), we can straightforwardly derive the conclusion that

$$\kappa(x,s,\delta_1)\le\frac{(\ln x)^{\delta_1-1}}{\Gamma(\delta_1)},\ \kappa(x,s,\delta_2)\le\frac{(\ln x)^{\delta_2-1}}{\Gamma(\delta_2)},\ (x,s)\in J\times J,$$

and then

$$K_1(x,s)\le\Big[\frac{1}{\Gamma(\delta_1)}+\frac{1}{\Lambda_1}\sum_{i=1}^{q_1}\nu_i\kappa(\varrho_1,s,\delta_1+\beta_i)+\frac{1}{\Lambda_1}\sum_{j=1}^{p_1}b_1\lambda_j\kappa(\zeta_j,s,\delta_1)\Big](\ln x)^{\delta_1-1}$$

$$= M_1(\ln x)^{\delta_1-1},\ (x,s)\in J\times J,$$

$$K_2(x,s) \le \Big[\frac{1}{\Gamma(\delta_2)} + \frac{1}{\Lambda_2}\sum_{\varsigma=1}^{p_1}\mu_\varsigma\kappa(\varrho_2,\varrho_2,\delta_2+\alpha_\varsigma) + \frac{1}{\Lambda_2}\sum_{\tau=1}^{p_2} b_2\pi_\tau\kappa(\eta_\tau,\eta_\tau,\delta_2)\Big](\ln x)^{\delta_2-1}$$

$$= M_2(\ln x)^{\delta_2-1},\ (x,s)\in J\times J,$$

thus, property (D1) is satisfied. Upon examining the form of the integral kernels $K_i^*(x,s), i=1,2$ as defined in (2.5), it becomes apparent that property (D2) is satisfied. □

Now we proceed to define a cone $N\subset U\times V$ by

$$N=\{(u,v)\in U\times V | u(x)\ge 0, v(x)\ge 0, {}^HD^{\delta_1-1}u(x)\ge 0, {}^HD^{\delta_2-1}v(x)\ge 0, x\in J\},$$

and then define the operator $\Phi: N\to U\times V$ as $\Phi(u,v)(x)=(\Phi_1(u,v)(x),\Phi_2(u,v)(x))$ for all $x\in J$, where the mappings $\Phi_1: N\to U\times V$ and $\Phi_2: N\to U\times V$ are given by

$$\begin{pmatrix}\Phi_1(u,v)(x)\\ \Phi_2(u,v)(x)\end{pmatrix} = \begin{pmatrix}\displaystyle\int_1^{+\infty} K_1(x,s)H_{1(u,v)}(s)\frac{\mathrm{d}s}{s}\\ \displaystyle\int_1^{+\infty} K_2(x,s)H_{2(u,v)}(s)\frac{\mathrm{d}s}{s}\end{pmatrix}, \tag{2.8}$$

with

$$H_{1(u,v)}(s) =: e_1(s,u(s),v(s),{}^HD^{\delta_1-1}u(s),{}^HD^{\delta_2-1}v(s)),$$
$$H_{2(u,v)}(s) =: e_2(s,u(s),v(s),{}^HD^{\delta_1-1}u(s),{}^HD^{\delta_2-1}v(s)).$$

In virtue of Lemma 2.3 and (2.8), for $(u,v)\in N, x\in J$, we have

$$\begin{pmatrix}{}^HD^{\delta_1-1}\Phi_1(u,v)(x)\\ {}^HD^{\delta_2-1}\Phi_2(u,v)(x)\end{pmatrix} = \begin{pmatrix}\displaystyle\int_1^{+\infty} K_1^*(x,s)H_{1(u,v)}(s)\frac{\mathrm{d}s}{s}\\ \displaystyle\int_1^{+\infty} K_2^*(x,s)H_{2(u,v)}(s)\frac{\mathrm{d}s}{s}\end{pmatrix}. \tag{2.9}$$

Throughout the present study, we make the fundamental assumption that the parameters e_1, e_2 satisfy the following conditions:

(F1) The functions $e_1, e_2\in C(J\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R},\mathbb{R}_+)$ and the constants $\Lambda_1,\Lambda_2>0$.

(F2) The nonnegative functions $a_i(s), b_i(s)(i=0,1,2,3,4)$ are confirmed to be integrable on the interval J and the nonnegative constants satisfying $0\le\theta_k,\vartheta_k<1(k=1,2,3,4)$ also meet

$$|e_1(s,u,v,l,m)|\le a_0(s)+a_1(s)|u|^{\theta_1}+a_2(s)|v|^{\theta_2}+a_3(s)|l|^{\theta_3}+a_4(s)|m|^{\theta_4},\forall s\in J,$$

with

$$\int_1^{+\infty} a_0(s)\frac{\mathrm{d}s}{s} = a_0^* < +\infty,$$
$$\int_1^{+\infty} a_1(s)[1+(\ln s)^{\delta_1-1}+(\ln s)^{\delta_2-1}]^{\theta_1}\frac{\mathrm{d}s}{s} = a_1^* < +\infty,$$
$$\int_1^{+\infty} a_2(s)[1+(\ln s)^{\delta_1-1}+(\ln s)^{\delta_2-1}]^{\theta_2}\frac{\mathrm{d}s}{s} = a_2^* < +\infty,$$

$$\int_1^{+\infty} a_3(s)\frac{\mathrm{d}s}{s} = a_3^* < +\infty,\ \int_1^{+\infty} a_4(t)\frac{\mathrm{d}s}{s} = a_4^* < +\infty,$$

and

$$|e_2(s,u,v,l,m)| \le b_0(s) + b_1(s)|u|^{\vartheta_1} + b_2(s)|v|^{\vartheta_2} + b_3(s)|l|^{\vartheta_3} + b_4(s)|m|^{\vartheta_4},\ \forall s \in J,$$

with

$$\begin{aligned}
&\int_1^{+\infty} b_0(s)\frac{\mathrm{d}s}{s} = b_0^* < +\infty,\\
&\int_1^{+\infty} b_1(s)[1 + (\ln s)^{\delta_1-1} + (\ln s)^{\delta_2-1}]^{\vartheta_1}\frac{\mathrm{d}s}{s} = b_1^* < +\infty,\\
&\int_1^{+\infty} b_2(s)[1 + (\ln s)^{\delta_1-1} + (\ln s)^{\delta_2-1}]^{\vartheta_2}\frac{\mathrm{d}s}{s} = b_2^* < +\infty,\\
&\int_1^{+\infty} b_3(s)\frac{\mathrm{d}s}{s} = b_3^* < +\infty,\ \int_1^{+\infty} b_4(s)\frac{\mathrm{d}s}{s} = b_4^* < +\infty.
\end{aligned}$$

(F3) The functions $e_1(s,u,v,l,m)$ and $e_2(s,u,v,l,m)$ exhibit a monotonic increasing behavior with regard to the variables u,v,l,m, and $e_1(s,0,0,0,0) \not\equiv 0, e_2(s,0,0,0,0) \not\equiv 0$ for all $s \in J$.

Lemma 2.6. *If assumptions (F1) and (F2) hold, then*

$$\int_1^{+\infty} |H_{1(u,v)}(s)|\frac{\mathrm{d}s}{s} \le a_0^* + \sum_{k=1}^{4} a_k^* ||(u,v)||_{U\times V}^{\vartheta_k},\ (u,v) \in U \times V, \tag{2.10}$$

and

$$\int_1^{+\infty} |H_{2(u,v)}(s)|\frac{\mathrm{d}s}{s} \le b_0^* + \sum_{k=1}^{4} b_k^* ||(u,v)||_{U\times V}^{\vartheta_k},\ (u,v) \in U \times V. \tag{2.11}$$

Proof. From assumptions (F1) and (F2), we obtain

$$\begin{aligned}
&\int_1^{+\infty} |H_{1(u,v)}(s)|\frac{\mathrm{d}s}{s}\\
&\le \int_1^{+\infty} \Big(a_0(s) + a_1(s)|u(s)|^{\theta_1} + a_2(s)|v(s)|^{\theta_2} + a_3(s)|{}^H D^{\delta_1-1}u(s)|^{\theta_3}\\
&\quad + a_4(s)|{}^H D^{\delta_2-1}v(s)|^{\theta_4}\Big)\frac{\mathrm{d}s}{s}\\
&\le a_0^* + \int_1^{+\infty} a_1(s)[1 + (\ln s)^{\delta_1-1} + (\ln s)^{\delta_2-1}]^{\theta_1}\frac{|u(s)|^{\theta_1}}{[1 + (\ln s)^{\delta_1-1} + (\ln s)^{\delta_2-1}]^{\theta_1}}\frac{\mathrm{d}s}{s}\\
&\quad + \int_1^{+\infty} a_2(s)[1 + (\ln s)^{\delta_1-1} + (\ln s)^{\delta_2-1}]^{\theta_2}\frac{|v(s)|^{\theta_2}}{[1 + (\ln s)^{\delta_1-1} + (\ln s)^{\delta_2-1}]^{\theta_2}}\frac{\mathrm{d}s}{s}\\
&\quad + \int_1^{+\infty} a_3(s)|{}^H D^{\delta_1-1}u(s)|^{\theta_3}\frac{\mathrm{d}s}{s} + \int_1^{+\infty} a_4(s)|{}^H D^{\delta_2-1}v(s)|^{\theta_4}\frac{\mathrm{d}s}{s}\\
&\le a_0^* + a_1^*||u||_U^{\theta_1} + a_2^*||v||_V^{\theta_2} + a_3^*||u||_U^{\theta_3} + a_4^*||v||_Y^{\theta_4}
\end{aligned}$$

$$\leq a_0^* + \sum_{k=1}^{4} a_k^* ||(u,v)||_{U\times V}^{\vartheta_k}\ ,\ (u,v) \in U \times V,$$

and

$$\begin{aligned}
&\int_1^{+\infty} |H_{2(u,v)}(s)| \frac{\mathrm{d}s}{s} \\
\leq &\int_1^{+\infty} \Big(b_0(s) + b_1(s)|u(s)|^{\vartheta_1} + b_2(s)|v(s)|^{\vartheta_2} + b_3(s)|{}^H D^{\delta_1 - 1} u(s)|^{\vartheta_3} \\
&+ b_4(s)|{}^H D^{\delta_2 - 1} v(s)|^{\vartheta_4} \Big) \frac{\mathrm{d}s}{s} \\
\leq &b_0^* + \int_1^{+\infty} b_1(s))[1 + (\ln s)^{\delta_1 - 1} + (\ln s)^{\delta_2 - 1}]^{\vartheta_1} \frac{|u(s)|^{\vartheta_1}}{[1 + (\ln s)^{\delta_1 - 1} + (\ln s)^{\delta_2 - 1}]^{\vartheta_1}} \frac{\mathrm{d}s}{s} \\
&+ \int_1^{+\infty} b_2(s))[1 + (\ln s)^{\delta_1 - 1} + (\ln s)^{\delta_2 - 1}]^{\vartheta_2} \frac{|v(s)|^{\vartheta_2}}{[1 + (\ln s)^{\delta_1 - 1} + (\ln s)^{\delta_2 - 1}]^{\vartheta_2}} \frac{\mathrm{d}s}{s} \\
&+ \int_1^{+\infty} b_3(s)|{}^H D^{\delta_1 - 1} u(s)|^{\vartheta_3} \frac{\mathrm{d}s}{s} + \int_1^{+\infty} b_4(s)|{}^H D^{\delta_2 - 1} v(s)|^{\vartheta_4} \frac{\mathrm{d}s}{s} \\
\leq &b_0^* + b_1^* ||u||_U^{\vartheta_1} + b_2^* ||v||_V^{\vartheta_2} + b_3^* ||u||_U^{\vartheta_3} + b_4^* ||v||_V^{\vartheta_4} \\
\leq &b_0^* + \sum_{k=1}^{4} b_k^* ||(u,v)||_{U\times V}^{\vartheta_k},\ (u,v) \in U \times V.
\end{aligned}$$

Thus lemma 2.6 holds. □

Lemma 2.7. (See [15]) *Let $\phi \subset \Omega$ denote a bounded subset,ϕ is relative compactness in Ω if subsequent requirements are met:*

(i) *For arbitrary $u \in \phi$, the functions $\frac{u(x)}{1+(\ln x)^{p-1}+(\ln x)^{q-1}}$ and ${}^H D^{p-1} u(x)$ are equicontinuous on any compact interval of J.*

(ii) *For arbitrary $\varepsilon > 0$ and $u \in \phi$, there exists a positive constant C depending only on ε such that*

$$\left| \frac{u(x_1)}{1 + (\ln x_1)^{p-1} + (\ln x_2)^{q-1}} - \frac{u(x_2)}{1 + (\ln x_2)^{p-1} + (\ln x_2)^{q-1}} \right| < \varepsilon$$

and $|{}^H D^{p-1} u(x_1) - {}^H D^{p-1} u(x_2)| < \varepsilon$ for any $x_1, x_2 \geq C$.

Lemma 2.8. *If assumptions (F1) and (F2) are fulfilled, then the operator $\Phi : N \to N$ is completely continuous.*

Proof. We divide the entire proof into four steps.

Step 1. Since $K_i(x,s), K_i^*(x,s)$ and f_i are nonnegative functions for $i = 1, 2$, it follows that $\Phi_1(u,v)(x)$, $\Phi_2(u,v)(x)$, ${}^H D^{\delta_1 - 1}\Phi_1(u,v)(x)$, ${}^H D^{\delta_2 - 1}\Phi_2(u,v)(x)$ are also nonnegative functions for any $(u,v) \in N,\ x \in J$, thus $\Phi : N \to N$.

Take $M = \{(u,v) | (u,v) \in N, ||(u,v)||_{U\times V} \leq \Delta\}$, where $\Delta > 0$ is a certain constant. By applying Lemma 2.4 in conjunction with equations (2.8), and (2.10), we obtain

$$\begin{aligned}
\sup_{x\in J} \frac{|\Phi_1(u,v)(x)|}{1 + (\ln x)^{\delta_1 - 1} + (\ln x)^{\delta_2 - 1}} \leq &\sup_{x\in J} \left| \int_1^{+\infty} \frac{K_1(x,s)}{1 + (\ln x)^{\delta_1 - 1} + (\ln x)^{\delta_2 - 1}} H_{1(u,v)}(s) \frac{\mathrm{d}s}{s} \right| \\
\leq &M_1 \int_1^{+\infty} |H_{1(u,v)}(s)| \frac{\mathrm{d}s}{s}
\end{aligned}$$

$$\leq M_1\Big(a_0^* + \sum_{k=1}^{4} a_k^* ||((u,v)||_{U\times V}^{\theta_k}\Big)$$
$$\leq M_1\Big(a_0^* + \sum_{k=1}^{4} a_k^* \Delta^{\theta_k}\Big),\ \forall\ (u,v) \in M, \tag{2.12}$$

and using Lemma 2.5, (2.9) and (2.10), we get

$$\begin{aligned}\sup_{x\in J} |{}^H D^{\delta_1-1}\Phi_1(u,v)(x)| \leq& \sup_{x\in J}\Big|\int_1^{\infty} K_1^*(x,s) H_{1(u,v)}(s)\frac{\mathrm{d}s}{s}\Big| \\ \leq& N_1 \int_1^{+\infty} |H_{1(u,v)}(s)|\frac{\mathrm{d}s}{s} \\ \leq& N_1\Big[a_0^* + \sum_{k=1}^{4} a_k^* ||(u,v)||_{U\times V}^{\theta_k}\Big] \\ \leq& N_1\Big[a_0^* + \sum_{k=1}^{4} a_k^* \Delta^{\theta_k}\Big],\ \forall\ (u,v) \in M. \end{aligned} \tag{2.13}$$

That is

$$\begin{aligned}||\Phi_1(u,v)||_U &= \max\Big\{\sup_{t\in J}\frac{|\Phi_1(u,v)(x)|}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}, \sup_{t\in J}|{}^H D^{\delta_1-1}\Phi_1(u,v)(x)|\Big\} \\ &\leq (M_1+N_1)\Big[a_0^* + \sum_{k=1}^{4} a_k^* \Delta^{\theta_k}\Big]. \end{aligned} \tag{2.14}$$

Similarly

$$||\Phi_2(u,v)||_V \leq (M_2+N_2)\Big[b_0^* + \sum_{k=1}^{4} b_k^* \Delta^{\vartheta_k}\Big], \forall\ (u,v) \in M.$$

Therefore

$$\begin{aligned}||\Phi(u,v)||_{U\times V} =& \max\Big\{\|\Phi_1(u,v)\|_U, \|\Phi_2(u,v)\|_V\Big\} \\ \leq& \Big(M_1+N_1+M_2+N_2\Big)\Big[a_0^* + b_0^* + \sum_{k=1}^{4}\big(a_k^*\Delta^{\theta_k} + b_k^*\Delta^{\vartheta_k}\big)\Big], \end{aligned} \tag{2.15}$$

so ΦM is uniformly bounded with respect to M.

Step 2. We proceed to establish the continuity of the mapping $\Phi : N \to N$. Set sequence $(u_n, v_n), (u, v) \in N$, such that (u_n, v_n) approaches (u, v) as $n \to \infty$. Thus $||(u_n, v_n)||_{U\times V} < +\infty, ||(u, v)||_{U\times V} < +\infty$. Similar to (2.12) and (2.13), we have

$$\sup_{x\in J}\frac{|\Phi_1(u_n,v_n)(x)|}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}} \leq M_1\Big[a_0^* + \sum_{k=1}^{4} a_k^* ||(u_n,v_n)||_{U\times V}^{\theta_k}\Big] < +\infty,$$

and

$$\sup_{x\in J}|{}^H D^{\delta_1-1}\Phi_1(u_n,v_n)(x)| \leq N_1\Big[a_0^* + \sum_{k=1}^{4} a_k^* ||(u_n,v_n)||_{U\times V}^{\theta_k}\Big] < +\infty.$$

From the continuity of the functions e_1, e_2, it follows that

$$\begin{aligned}\lim_{n\to\infty}\frac{\Phi_1(u_n,v_n)(x)}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}&=\lim_{n\to\infty}\int_1^{+\infty}\frac{K_1(x,s)}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}H_{1(u_n,v_n)}(s)\frac{\mathrm{d}s}{s}\\&=\int_1^{+\infty}\frac{K_1(x,s)}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}H_{1(u,v)}(s)\frac{\mathrm{d}s}{s}\\&=\frac{\Phi_1(u,v)(x)}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}},\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}{}^{H}D^{\delta_1-1}\Phi_1(u_n,v_n)(x)&=\lim_{n\to\infty}\int_1^{+\infty}K_1^*(x,s)H_{1(u_n,v_n)}(s)\frac{\mathrm{d}s}{s}\\&=\int_1^{\infty}K_1^*(x,s)H_{1(u,v)}(s)\frac{\mathrm{d}s}{s}\\&={}^{H}D^{\delta_1-1}\Phi_1(u,v)(x).\end{aligned}$$

Noticed that

$$\begin{aligned}&\big|H_{1(u_n,v_n)}(s)-H_{1(u,v)}(s)\big|\\\le&|H_{1(u_n,v_n)}(s)|+|H_{1(u,v)}(s)|\\\le&2a_0(s)+a_1(s))[1+(\ln s)^{\delta_1-1}+(\ln s)^{\delta_2-1}]^{\theta_1}\big(||(u_n,v_n)||_{U\times V}^{\theta_1}+||(u,v)||_{U\times V}^{\theta_1}\big)\\&+a_2(s))[1+(\ln s)^{\delta_1-1}+(\ln s)^{\delta_2-1}]^{\theta_2}\big(||(u_n,v_n)||_{U\times V}^{\theta_2}+||(u,v)||_{U\times V}^{\theta_2}\big)\\&+a_3(s)\big(||(u_n,v_n)||_{U\times V}^{\theta_3}+||(u,v)||_{U\times V}^{\theta_3}\big)\\&+a_4(s)\big(||(u_n,v_n)||_{U\times V}^{\theta_4}+||(u,v)||_{U\times V}^{\theta_4}\big).\end{aligned}$$

By the Lebesgue dominated convergence theorem, we are able to derive

$$\begin{aligned}&\sup_{x\in J}\frac{|\Phi_1(u_n,v_n)(x)-\Phi_1(u,v)(x)|}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}\\\le&\sup_{x\in J}\int_1^{+\infty}\frac{K_1(x,s)}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}\big|H_{1(u_n,v_n)}(s)-H_{1(u,v)}(s)\big|\frac{\mathrm{d}s}{s}\\\le&M_1\Big[\int_1^{+\infty}\big|H_{1(u_n,v_n)}(s)-H_{1((u,v)}(s)\big|\frac{\mathrm{d}s}{s}\Big]\\\to&0,\quad\text{as}\quad n\to\infty,\end{aligned}$$

and

$$\begin{aligned}&\sup_{x\in J}|{}^{H}D^{\delta_1-1}\Phi_1(u_n,v_n)(x)-{}^{H}D^{\delta_1-1}\Phi_1(u,v)(x)|\\\le&\sup_{x\in J}\int_1^{+\infty}K_1^*(x,s)\big|H_{1(u_n,v_n)}(s)-H_{1(u,v)}(s)\big|\frac{\mathrm{d}s}{s}\\\le&N_1\int_1^{+\infty}\big|H_{1(u_n,v_n)}(s)-H_{1((u,v)}(s)\big|\frac{\mathrm{d}s}{s}\\\to&0,\quad\text{as}\quad n\to\infty.\end{aligned}$$

Thus

$$\begin{aligned}
&\|\Phi_1(u_n,v_n)-\Phi_1(u,v)\|_U\\
&=\max\Big\{\sup_{x\in J}\frac{|\Phi_1(u_n,v_n)(x)-\Phi_1(u,v)(x)|}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}},\\
&\quad\sup_{x\in J}|{}^H\!D^{\delta_1-1}\Phi_1(u_n,v_n)(x)-{}^H\!D^{\delta_1-1}\Phi_1(u,v)(x)|\Big\}\\
&\to 0,\ n\to\infty,
\end{aligned}$$

so we conclude that Φ_1 is continuous. Similarly, Φ_2 is also continuous. The combined function Φ, which is a composition of Φ_1 and Φ_2, therefore inherits this property and is itself continuous.

Step 3. Let $\Xi\subset J$ denote an arbitrary compact subinterval. For any $u_1,u_2\in\Xi$ satisfying $u_2>u_1$, we have

$$\begin{aligned}
&\Big|\frac{\Phi_1(u,v)(u_2)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{\Phi_1(u,v)(u_1)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\Big|\\
&\le\int_1^{+\infty}\Big|\frac{K_1(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_1(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\Big|\big|H_{1(u,v)}(s)\big|\frac{\mathrm{d}s}{s},\ (u,v)\in M.
\end{aligned}\tag{2.16}$$

Direct observation clearly indicates that the function

$$\frac{K_1(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_1(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}$$

is uniformly continuous for any $(u_1,s),(u_2,s)\in\Xi\times\Xi$. In reality, for a fixed $s\in\Xi$ and $s\le x$, we have

$$\begin{aligned}
&\frac{K_1(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_1(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\\
=&\frac{g(u_2,s,\delta_1)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}+\frac{(\ln u_2)^{\delta_1-1}}{\Lambda_1[1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}]}\\
&\times\Big[\sum_{i=1}^{q_1}\nu_i g(\varrho_1,s,\delta_1+\beta_i)+\sum_{j=1}^{p_1}b_1\lambda_j g(\zeta_j,s,\delta_1)\Big]-\frac{g(u_1,s,\delta_1)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\\
&-\frac{(\ln u_1)^{\delta_1-1}}{\Lambda_1[1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}]}\times\Big[\sum_{i=1}^{q_1}\nu_i g(\varrho_1,s,\delta_1+\beta_i)+\sum_{j=1}^{p_1}b_1\lambda_j g(\zeta_j,s,\delta_1)\Big]\\
=&\frac{(\ln u_2)^{\delta_1-1}-(\ln u_2-\ln s)^{\delta_1-1}}{\Gamma(\delta_1)[1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}]}-\frac{(\ln u_1)^{\delta_1-1}-(\ln u_1-\ln s)^{\delta_1-1}}{\Gamma(\delta_1)[1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}]}\\
&+\Big\{\frac{(\ln u_2)^{\delta_1-1}}{\Lambda_1[1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}]}-\frac{(\ln u_1)^{\delta_1-1}}{\Lambda_1[1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}]}\Big\}\\
&\times\Big\{\sum_{i=1}^{q_1}\nu_i\Big[(\ln\varrho_1)^{\delta_1+\beta_i-1}-(\ln\varrho_1-\ln s)^{\delta_1+\beta_i-1}\Big]+\sum_{j=1}^{p_1}b_1\lambda_j\Big[(\ln\zeta_j)^{\delta_1-1}-(\ln\zeta_j-\ln s)^{\delta_1-1}\Big]\Big\}\\
\to&0,\ \text{as } u_2\to u_1.
\end{aligned}$$

Then the function

$$\frac{K_1(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_1(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}$$

is continuous on the interval Ξ, thus it is also uniformly continuous.

Furthermore, for a fixed $s\in\Xi$ such that $s\geq x$, the function

$$\begin{aligned}
&\frac{K_1(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_1(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\\
=&\frac{(\ln u_2)^{\delta_1-1}}{\Gamma(\delta_1)[1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}]}-\frac{(\ln u_1)^{\delta_1-1}}{\Gamma(\delta_1)[1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}]}\\
&+\Big\{\frac{(\ln u_2)^{\delta_1-1}}{\Lambda_1[1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}]}-\frac{(\ln u_1)^{\delta_1-1}}{\Lambda_1[1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}]}\Big\}\\
&\times\Big[\sum_{i=1}^{q_1}\nu_i(\ln\varrho_1)^{\delta_1+\beta_i-1}+\sum_{j=1}^{p_1}b_1\lambda_j(\ln\zeta_j)^{\delta_1-1}\Big]
\end{aligned}$$

is independent of s, then the function

$$\frac{K_1(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_1(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}$$

is also uniformly continuous on Ξ. Noting that this independence from s holds for all $s\in J/\Xi$ with $s\geq x$, the function is consequently uniformly continuous on J.

By the same argument, we can show that for all $s\in J$ and $u_1,u_2\in\Xi$, the function

$$\frac{K_2(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_2(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}$$

exhibits uniform continuity over the interval J.

Consequently, for any given $s\in J$ and $u_1,u_2\in\Xi$, the following holds

$$\begin{aligned}
&\forall\epsilon>0,\exists\delta(\epsilon)\ \text{ such that if }\ |u_1-u_2|<\delta,\ \text{ then}\\
&\Big|\frac{K_1(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_1(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\Big|<\epsilon,\\
&\Big|\frac{K_2(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_2(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\Big|<\epsilon.
\end{aligned}\tag{2.17}$$

Combine (2.10), (2.11), (2.16) and (2.17) yields

$$\begin{aligned}
&\Big|\frac{\Phi_1(u,v)(u_2)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{\Phi_1(u,v)(u_1)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\Big|\leq\Big[a_0^*+\sum_{k=1}^{4}a_k^*\Delta^{\theta_k}\Big]\epsilon,\\
&\Big|\frac{\Phi_2(u,v)(u_2)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{\Phi_2(u,v)(u_1)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\Big|\leq\Big[b_0^*+\sum_{k=1}^{4}b_k^*\Delta^{\vartheta_k}\Big]\epsilon,
\end{aligned}$$

for all $s\in J$, $(u,v)\in U$ and $u_1,u_2\in\Xi$. Therefore the functions $\Phi_1(u,v)(x)/(1+(\ln x)^{\delta_1-1}+(\ln tx)^{\delta_2-1})$ and $\Phi_2(u,v)(x)/(1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1})$ are equicontinuous on Ξ.

Notice that

$$
\begin{aligned}
{}^{H}D^{\delta_1-1}\Phi_1(u,v)(x) &= \int_1^{+\infty} K_1^*(x,s)H_{1(u,v)}(s)\frac{\mathrm{d}s}{s},\\
{}^{H}D^{\delta_2-1}\Phi_2(u,v)(x) &= \int_1^{+\infty} K_2^*(x,s)H_{2(u,v)}(s)\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

Using the same method as above and the representational forms of the kernel functions $K_1^*(x,s), K_2^*(x,s)$, we can establish that the fractional derivative expressions ${}^{H}D^{\delta_1-1}\Phi_1(u,v)(x)$, ${}^{H}D^{\delta_2-1}\Phi_2(u,v)(x)$ are also equicontinuous on Ξ. So the first condition of Lemma 2.7 is fulfilled.

Step 4. We demonstrate that the operators Φ_1 and Φ_2 are equiconvergent as $x \to +\infty$. Due to

$$
\lim_{x\to+\infty}\frac{K_1(x,s)}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}=0,\ \lim_{x\to+\infty}\frac{K_2(x,s)}{1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1}}=0,
$$

then for any positive value ϵ, there exists a sufficiently large constant $C = C(\epsilon) > 0$ such that for any $u_1, u_2 \geq C$ and $s \in J$, we have

$$
\begin{aligned}
&\Big|\frac{K_1(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_1(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\Big|<\epsilon,\\
&\Big|\frac{K_2(u_2,s)}{1+(\ln u_2)^{\delta_1-1}+(\ln u_2)^{\delta_2-1}}-\frac{K_2(u_1,s)}{1+(\ln u_1)^{\delta_1-1}+(\ln u_1)^{\delta_2-1}}\Big|<\epsilon.
\end{aligned}
$$

It is easily shown with the help of Lemma 2.6 and (2.16) that $\Phi_1(u,v)(x)/(1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1})$ and $\Phi_2(u,v)(x)/(1+(\ln x)^{\delta_1-1}+(\ln x)^{\delta_2-1})$ are aequiconvergent at $+\infty$. Furthermore, from the form of $K_1^*(x,s)$ and $K_2^*(x,s)$, it can be obtained that ${}^{H}D^{\delta_1-1}\Phi_1(u,v)(x)$ and ${}^{H}D^{\delta_2-1}\Phi_2(u,v)(x)$ are equiconvergent at $+\infty$. Thus, the second condition of Lemma 2.7 holds.

From the foregoing results, it follows that Φ is uniformly bounded and relatively compact in N. Consequently, $\Phi : N \to N$ is completely continuous. This completes the proof of Lemma 2.8. □

3. Main results and applications

In this work, a novel partial order relation is formulated with its formal definition presented as follows:

$$
\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \geq \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}
$$

if $u_1(x) \geq u_2(x)$, $v_1(x) \geq v_2(x)$, ${}^{H}D^{\delta_1-1}u_1(x) \geq {}^{H}D^{\delta_1-1}u_2(x)$, ${}^{H}D^{\delta_2-1}v_1(x) \geq D^{\delta_2-1}v_2(x)$, $x \in J$.

Theorem 3.1. *If conditions (F1), (F2) and (F3) are satisfied, then the HFDS* (1.1) *possesses two sets of approximate sequences of solution* (u_n, v_n) *and* (l_n, m_n) *that converge to the positive extreme solutions* (u^*, v^*) *and* (l^*, m^*) *equipped with*

$$
0 \leq \|(u^*, v^*)\|_{U\times V} \leq \Upsilon \text{ and } 0 \leq \|(l^*, m^*)\|_{U\times V} \leq \Upsilon,
$$

where Υ is a prescribed real constant. The sequences (u_n, v_n) and (l_n, m_n) may be constructed according to the following specifications:

$$\begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} = \begin{pmatrix} \Phi_1(u_{n-1}, v_{n-1})(x) \\ \Phi_2(u_{n-1}, v_{n-1})(x) \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} = \begin{pmatrix} \Upsilon(\ln x)^{\delta_1} \\ \Upsilon(\ln x)^{\delta_2} \end{pmatrix} \tag{3.1}$$

and

$$\begin{pmatrix} l_n(x) \\ m_n(x) \end{pmatrix} = \begin{pmatrix} \Phi_1(l_{n-1}, m_{n-1})(x) \\ \Phi_2(l_{n-1}, m_{n-1})(x) \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} l_0(x) \\ m_0(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.2}$$

for $n = 1, 2, \cdots$. Moreover, there are two distinct groups of sequences that demonstrate the following monotonic properties:

$$\begin{aligned} \begin{pmatrix} l_0(x) \\ m_0(x) \end{pmatrix} \leq \begin{pmatrix} l_1(x) \\ m_1(x) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} l_n(x) \\ m_n(x) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} l^*(x) \\ m^*(x) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} u^*(x) \\ v^*(x) \end{pmatrix} \\ \leq \cdots \leq \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} \leq \cdots \leq \begin{pmatrix} u_2(x) \\ v_2(x) \end{pmatrix} \leq \begin{pmatrix} u_1(x) \\ v_1(x) \end{pmatrix} \leq \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}. \end{aligned} \tag{3.3}$$

Proof. By Lemma 2.2, $(u, v) \in M$ is a positive solution of the HFDS (1.1) if and only if it is a positive fixed point of the operator Φ. Therefore, the problem reduces to finding the fixed points of the operator Φ.

Lemma 2.8 establishes that $\Phi(M) \subset M$. Let

$$\Upsilon \geq \max\Big\{5\Lambda a_0^*, 5\Lambda b_0^*, (5\Lambda a_k^*)^{1/(1-\theta_k)}, (5\Lambda b_k^*)^{1/(1-\vartheta_k)}, k = 1, 2, 3, 4\Big\}, \tag{3.4}$$

where $\Lambda = \max\Big\{M_1 + M_2, N_1 + N_2\Big\}$.

Take $M_\Upsilon = \{(u, v) \in P : ||(u, v)||_{U\times V} \leq \Upsilon\}$. For every pair $(u, v) \in M_\Upsilon$, analogous to the approach in (2.12) and (2.13), direct calculation yields

$$\sup_{t\in J} \frac{|\Phi_1(u, v)(x)|}{1 + (\ln x)^{\delta_1 - 1} + (\ln x)^{\delta_2 - 1}} \leq M_1\Big[a_0^* + \sum_{k=1}^{4} a_k^* \Upsilon^{\theta_k}\Big] \leq \Upsilon$$

and

$$\sup_{xt\in J} |{}^H D^{\delta_1 - 1} u(x)| \leq N_1\Big[a_0^* + \sum_{k=1}^{4} a_k^* \Upsilon^{\theta_k}\Big] \leq \Upsilon,$$

which indicates that $||\Phi_1(u, v)||_U \leq \Upsilon$. In the same way, $||\Phi_2(u, v)||_V \leq \Upsilon$ for all $((u, v) \in M_\Upsilon$. Therefore we obtain

$$||\Phi(u, v)||_{U\times V} = \Big\{\|\Phi_1(u, v)\|_U, \|\Phi_2(u, v)\|_V\Big\} \leq \Upsilon, \ \forall (u, v) \in M_\Upsilon.$$

That is, $\Phi(M_\Upsilon) \subset M_\Upsilon$.

Base on equations (3.1) and (3.2), we can derive that$(u_0(x), v_0(x)), (l_0(x), m_0(x)) \in M_\Upsilon$. Here we introduce two families of sequences (u_n, v_n) and (l_n, m_n), where the recursive relations are defined as $(u_n, v_n) = \Phi(u_{n-1}, v_{n-1}), (l_n, m_n) = \Phi(l_{n-1}, m_{n-1})$ for all positive integers $n = 1, 2, \ldots$. From the inclusion property $\Phi(M_\Upsilon) \subset M_\Upsilon$, it can be further verified that the sequences (u_n, v_n) and (l_n, m_n) satisfy $(u_n, v_n), (l_n, m_n) \in \Phi(U_\Upsilon)$.

From (2.8), (3.1), along with conditions (D1) and (F2), for any $t \in J$, it follows that

$$u_1(x) = \Phi_1(u_0, v_0)(x) \le \Lambda\Big[a_0^* + \sum_{k=1}^{4} a_k^* \Upsilon^{\theta_k}\Big](\ln x)^{\delta_1 - 1} \le \Upsilon(\ln x)^{\delta_1 - 1} = u_0(x)$$

and

$$v_1(x) = \Phi_2(u_0, v_0)(x) \le \Lambda\Big[b_0^* + \sum_{k=1}^{4} b_k^* \Upsilon^{\vartheta_k}\Big](\ln x)^{\delta_2 - 1} \le \Upsilon(\ln x)^{\delta_2 - 1} = v_0(x),$$

therefore

$$u_1(x) \le u_0(x),\ v_1(x) \le v_0(x). \tag{3.5}$$

By (3.5), conditions (D2) and (F2), for any $t \in J$, we have

$$\begin{aligned}
{}^{H}D^{\delta_1 - 1}u_1(x) &= {}^{H}D^{\delta_1 - 1}\Phi_1(u_0, v_0)(x) \\
&= \int_0^{+\infty} K_1^*(x, s) H_{1(u_0, v_0)}(s) \frac{\mathrm{d}s}{s} \\
&\le \Lambda\Big[a_0^* + \sum_{k=1}^{4} a_k^* \Upsilon^{\theta_k}\Big] \\
&\le \Upsilon \\
&= {}^{H}D^{\delta_1 - 1}u_0(x), \\
{}^{H}D^{\delta_2 - 1}v_1(x) &= {}^{H}D^{\delta_2 - 1}\Phi_2(u_0, v_0)(x) \\
&= \int_0^{+\infty} K_2^*(x, s) H_{2(u_0, v_0)}(s) \frac{\mathrm{d}s}{s} \\
&\le \Lambda\Big[b_0^* + \sum_{k=1}^{4} b_k^* \Upsilon^{\vartheta_k}\Big] \\
&\le \Upsilon \\
&= {}^{H}D^{\delta_2 - 1}v_0(x),
\end{aligned}$$

that is

$${}^{H}D^{\delta_1 - 1}u_1(x) \le {}^{H}D^{\delta_1 - 1}u_0(x),\ {}^{H}D^{\delta_2 - 1}v_1(x) \le {}^{H}D^{\delta_2 - 1}v_0(x). \tag{3.6}$$

By (3.5), (3.6) and (F3), the aforementioned steps are repeated to perform the second iteration.

$$\begin{aligned}
u_2(x) &= \Phi_1(u_1, v_1)(x) \le \Phi_1(u_0, v_0)(x) = u_1(x), \\
v_2(x) &= \Phi_2(u_1, v_1)(x) \le \Phi_2(u_0, v_0)(x) = v_1(x),
\end{aligned}$$

for all $t \in J$, and

$$\begin{aligned}
{}^{H}D^{\delta_1 - 1}u_2(x) &= {}^{H}D^{\delta_1 - 1}\Phi_1(u_1, v_1)(x) \le {}^{H}D^{\delta_1 - 1}\Phi_1(u_0, v_0)(x) = {}^{H}D^{\delta_1 - 1}u_1(x), \\
{}^{H}D^{\delta_2 - 1}v_2(x) &= {}^{H}D^{\delta_2 - 1}\Phi_2(u_1, v_1)(x) \le {}^{H}D^{\delta_2 - 1}\Phi_2(u_0, v_0)(x) = {}^{H}D^{\delta_2 - 1}v_1(x).
\end{aligned}$$

For $x \in J$, iterating this process yields

$$u_n(x) \le u_{n-1}(x),\ v_n(t) \le v_{n-1}(x), \tag{3.7}$$

$${}^{H}D^{\delta_1 - 1}u_n(x) \le {}^{H}D^{\delta_1 - 1}u_{n-1}(x),\ {}^{H}D^{\delta_2 - 1}v_n(x) \le {}^{H}D^{\delta_2 - 1}v_{n-1}(x). \tag{3.8}$$

That is

$$\begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} \leq \begin{pmatrix} u_{n-1}(x) \\ v_{n-1}(x) \end{pmatrix}.$$

Furthermore, $\Phi : M_\Upsilon \to M_\Upsilon$ is completely continuous, which ensures the existence of a subsequence S of $\mathbf{N}$ such that $(u_n, v_n) \to (u^*, v^*) \in M_\Upsilon$. This result, together with (3.7) and (3.8), implies that $(u_n, v_n) \to (u^*, v^*)$ as $n \to \infty$. Considering the continuity property of mapping Φ along with the iterative relation $(u_{n+1}, v_{n+1}) = \Phi(u_n, v_n)$, we can consequently derive that $(u^*, v^*) = \Phi(u^*, v^*)$. Thus, it can be established that (u^*, v^*) represents a fixed point of the operator Φ.

For a set of sequence (l_n, m_n), an analogous argument yields

$$\begin{aligned} l_1(x) &= \Phi_1(l_0, m_0)(x) = \int_1^{+\infty} K_1(x,s) H_{1(l_0,m_0)}(s) \frac{\mathrm{d}s}{s} \geq 0 = l_0(x), \\ m_1(x) &= \Phi_2(l_0, m_0)(x) = \int_1^{+\infty} K_2(x,s) H_{2(l_0,m_0)}(s) \frac{\mathrm{d}s}{s} \geq 0 = m_0(x), \end{aligned}$$

and

$$\begin{aligned} {}^H D^{\delta_1 - 1} l_1(x) &= \int_1^{+\infty} K_1^*(x,s) H_{1(l_0,m_0)}(s) \frac{\mathrm{d}s}{s} \geq 0 = {}^H D^{\delta_1 - 1} l_0(x), \\ {}^H D^{\delta_2 - 1} m_1(x) &= \int_1^{+\infty} K_2^*(x,s) H_{2(l_0,m_0)}(s) \frac{\mathrm{d}s}{s} \geq 0 = {}^H D^{\delta_2 - 1} m_0(x). \end{aligned}$$

By (F3) we know

$$\begin{aligned} l_2(x) &= \int_1^{+\infty} K_1^*(x,s) H_{1(l_1,m_1)}(s) \frac{\mathrm{d}s}{s} \geq \Phi_1(l_0, m_0)(x) = l_1(x), \\ m_2(x) &= \int_1^{+\infty} K_1^*(x,s) H_{2(l_1,m_1)}(s) \frac{\mathrm{d}s}{s} \geq \Phi_2(l_0, m_0)(x) = m_1(x), \end{aligned}$$

and

$$\begin{aligned} {}^H D^{\delta_1 - 1} l_2(x) &= {}^H D^{\delta_1 - 1} \Phi_1(l_1, m_1)(x) \geq {}^H D^{\delta_1 - 1} \Phi_1(l_0, m_0)(x) = {}^H D^{\delta_1 - 1} l_1(x), \\ {}^H D^{\delta_2 - 1} m_2(x) &= {}^H D^{\delta_2 - 1} \Phi_2(l_1, m_1)(x) \geq {}^H D^{\delta_2 - 1} \Phi_2(l_0, m_0)(x) = {}^H D^{\delta_2 - 1} m_1(x). \end{aligned}$$

Similarly, for $n = 2, 3, \cdots$ and $t \in J$, repeated iteration yields

$$l_n(x) \geq l_{n-1}(x), \;\; m_n(x) \geq m_{n-1}(x),$$

and

$${}^H D^{\delta_1 - 1} l_n(x) \geq {}^H D^{\delta_1 - 1} l_{n-1}(x), \;\; {}^H D^{\delta_2 - 1} m_n(x) \geq {}^H D^{\delta_2 - 1} m_{n-1}(x).$$

That is

$$\begin{pmatrix} l_n(x) \\ m_n(x) \end{pmatrix} \geq \begin{pmatrix} l_{n-1}(x) \\ m_{n-1}(x) \end{pmatrix}.$$

Considering the continuity property of mapping Φ along with the iterative relation $(l_n, m_n) = \Phi(l_{n-1}, m_{n-1})$, we can derive that $(l_n, m_n) \to (l^*, m^*)$ and $\Phi(l^*, m^*) = (l^*, m^*)$. Thus, it can be obtained that (l^*, m^*) represents a fixed point of the operator Φ.

Finally, we establish that (u^*, v^*) and (l^*, m^*) are two distinct sets of extreme positive solutions for the system (1.1). Suppose that $(\alpha(x), \beta(x))$ constitutes a set of positive solutions to the system (1.1). It follows that

$$\Phi(\alpha(x), \beta(x)) = (\alpha(x), \beta(x)),$$

and

$$\begin{aligned} l_0(x) = 0 \le \alpha(x) \le \Upsilon(\ln x)^{\delta_1 - 1} = u_0(x), \\ m_0(x) = 0 \le \beta(x) \le \Upsilon(\ln x)^{\delta_2 - 1} = v_0(x), \end{aligned}$$

and

$$\begin{aligned} {}^H D^{\delta_1 - 1} l_0(x) = 0 \le {}^H D^{\delta_1 - 1} \alpha(x) \le \Upsilon = {}^H D^{\delta_1 - 1} u_0(x), \\ {}^H D^{\delta_2 - 1} m_0(x) = 0 \le {}^H D^{\delta_2 - 1} \beta(x) \le \Upsilon = {}^H D^{\delta_2 - 1} v_0(x). \end{aligned}$$

By the monotonicity of the operator Φ, we obtain

$$\begin{aligned} l_1(x) = \Phi_1(l_0, m_0)(x) \le \alpha(x) \le \Phi_1(u_0, v_0)(x) = u_1(x), \\ m_1(x) = \Phi_2(l_0, m_0)(x) \le \beta(x) \le \Phi_2(l_0, m_0)(x) = v_1(x), \end{aligned}$$

and

$$\begin{aligned} {}^H D^{\delta_1 - 1} l_1(x) \le {}^H D^{\delta_1 - 1} \alpha(x) \le {}^H D^{\delta_1 - 1} u_1(x), \\ {}^H D^{\delta_2 - 1} m_1(x) \le {}^H D^{\delta_2 - 1} \beta(x) \le {}^H D^{\delta_2 - 1} v_1(x). \end{aligned}$$

By repeating the above steps, we obtain

$$\begin{aligned} l_n(x) = \Phi_1(l_{n-1}, m_{n-1})(x) \le \alpha(x) \le \Phi_1(u_{n-1}, v_{n-1})(x) = u_n(x), \\ m_n(x) = \Phi_2(l_{n-1}, m_{n-1})(x) \le \beta(x) \le \Phi_2(l_{n-1}, m_{n-1})(x) = v_n(x), \end{aligned}$$

and

$$\begin{aligned} {}^H D^{\delta_1 - 1} l_n(x) \le {}^H D^{\delta_1 - 1} \alpha(x) \le {}^H D^{\delta_1 - 1} u_n(x), \\ {}^H D^{\delta_2 - 1} m_n(x) \le {}^H D^{\delta_2 - 1} \beta(x) \le {}^H D^{\delta_2 - 1} v_n(x). \end{aligned}$$

That is

$$\begin{pmatrix} l_n(x) \\ m_n(x) \end{pmatrix} \le \begin{pmatrix} \alpha(x) \\ \beta(x) \end{pmatrix} \le \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix}.$$

Owing to $\lim\limits_{n \to \infty} (l_n, m_n) = (l^*, m^*)$ and $\lim\limits_{n \to \infty} (u_n, v_n) = (u^*, v^*)$, the results presented in (3.3) hold.

Clearly, the system (1.1) has no zero solution since $e_1(x, 0, 0, 0, 0) \neq 0$ and $e_2(x, 0, 0, 0, 0) \neq 0$ for all $x \in J$. From (3.3), it is clear that (l^*, m^*) and (u^*, v^*) are two extreme positive solutions

of system (1.1) which are obtained by applying two different monotone iterative sequences in (3.1) and (3.2), respectively. This completes the proof. □

Remark 3.1. Assumption (F2) can be replaced by the following assumption:

(F4) The nonnegative functions $a_i(x), b_i(x)(i = 1, 2, 3, 4)$ are integrable in J and the nonnegative constants $\theta_k, \vartheta_k > 1(k = 1, 2, 3, 4)$ satisfy

$$|f_1(x, u, v, l, m)| \le a_1(x)|u|^{\theta_1} + a_2(x)|v|^{\theta_2} + a_3(x)|l|^{\theta_3} + a_4(x)|m|^{\theta_4}, \ \forall x \in J,$$
$$|f_2(x, u, v, l, m)| \le b_1(x)|u|^{\vartheta_1} + b_2(x)|v|^{\vartheta_2} + b_3(x)|l|^{\vartheta_3} + b_4(x)|m|^{\vartheta_4}, \ \forall x \in J.$$

If all other assumptions are consistent with those of (F2), then the conclusion of Theorem 3.1 remains valid, provided that the restriction on Υ in (3.4) is replaced by the following:

$$0 < \Upsilon < \min\Big\{(\frac{1}{4\Lambda a_k^*})^{\frac{1}{\theta_k - 1}}, (\frac{1}{4\Lambda b_k^*})^{\frac{1}{\vartheta_k - 1}}, k = 1, 2, 3, 4\Big\},$$

where $\Lambda = \min\{M_1 + M_2, N_1 + N_2\}$. The same results can then be obtained by arguments similar to those used in the proof of Theorem 3.1.

Example 3.1. Consider the HFDS defined on the half-line

$$\begin{cases} {}^HD^{5/2}u(x) + e_1(x, u(x), v(x), {}^HD^{\delta_1 - 1}u(x), {}^HD^{\delta_2 - 1}v(x)) = 0, \\ {}^HD^{5/2}v(x) + e_2(x, u(x), v(x), {}^HD^{\delta_1 - 1}u(x), {}^HD^{\delta_2 - 1}v(x)) = 0, \\ u(1) = u'(1) = 0, \ {}^HD^{3/2}u(+\infty) = \sum\limits_{i=1}^{2} \nu_i {}^HI^{\beta_i}u(e^{1/2}) + \dfrac{\Gamma(5/2)}{6}\sum\limits_{j=1}^{3}\lambda_j u(\zeta_j), \\ v(1) = v'(1) = 0, \ {}^HD^{3/2}v(+\infty) = \sum\limits_{\varsigma=1}^{3} \mu_\varsigma {}^HI^{\alpha_\varsigma}v(e) + \dfrac{\Gamma(5/2)}{14}\sum\limits_{\tau=1}^{3}\pi_\tau v(\eta_\tau). \end{cases} \tag{3.9}$$

Corresponding to the HFDS (1.1), where

$$\delta_1 = 5/2,\ n_1 = 3,\ q_1 = 2,\ p_1 = 3, \nu_1 = 1,\ \nu_2 = 2,\ \varrho_1 = e^{1/2},\ \beta_1 = \frac{1}{2},\ \beta_2 = \frac{3}{2},\ b_1 = \frac{\Gamma(\delta_1)}{6},$$
$$\lambda_1 = \frac{1}{4},\ \lambda_2 = \frac{3}{8\sqrt{2}},\ \lambda_3 = \frac{2}{3\sqrt{3}},\ \zeta_1 = e,\ \zeta_2 = e^2,\ \zeta_3 = e^3,$$
$$\delta_2 = 5/2,\ n_2 = 3,\ q_2 = 3,\ p_2 = 3,\ \mu_1 = \frac{1}{3},\ \mu_2 = \frac{3}{4},\ \mu_3 = 1,\ \varrho_2 = e,\ \alpha_1 = \frac{1}{2},\ \alpha_2 = \frac{3}{2},\ \alpha_3 = \frac{5}{2},$$
$$b_2 = \frac{\Gamma(\delta_2)}{14},\ \pi_1 = \frac{\sqrt{2}}{4},\ \pi_2 = \frac{2\sqrt{3}}{9},\ \pi_3 = \frac{1}{2},\ \eta_1 = e^2,\ \eta_2 = e^3,\ \eta_3 = e^4,$$
$$H_{1(u,v)} = \frac{2x}{(9+x)^2} + \frac{xe^{-x}|u(x)|^{0.1}}{[1 + (\ln x)^{1.5} + (\ln x)^{1.5}]^{0.1}} + \frac{xe^{-2x}|v(x)|^{0.3}}{[1 + (\ln x)^{1.5} + (\ln x)^{1.5}]^{0.3}}$$
$$+ xe^{-10x}|{}^HD^{1.5}u(x)|^{0.4} + \frac{x|{}^HD^{1.5}v(x)|^{0.1}}{1 + x^2},$$
$$H_{2(u,v)} = \frac{x}{20(1+x^2)} + \frac{xe^{-3x}|u(x)|^{0.2}}{[1 + (\ln x)^{1.5} + (\ln x)^{1.5}]^{0.2}} + \frac{xe^{-4x}|v(x)|^{0.4}}{[1 + (\ln x)^{1.5} + (\ln x)^{1.5}]^{0.4}}$$
$$+ \frac{3x^3|{}^HD^{1.5}u(x)|^{0.2}}{(3+x^3)^2} + \frac{x|{}^HD^{1.5}v(x)|^{0.3}}{10(1+x^2)}.$$

Through calculation, we obtained

$$\Lambda_1 = \Gamma(\delta_1) - \sum_{i=1}^{q_1} \nu_i \frac{\Gamma(\delta_1)}{\Gamma(\delta_1+\beta_i)} (\ln \varrho_1)^{\delta_1+\beta_i-1} - b_1 \sum_{j=1}^{p_1} \lambda_j (\ln \zeta_j)^{q_1-1} = \frac{3}{16}\sqrt{\pi} > 0,$$

$$\Lambda_2 = \Gamma(\delta_2) - \sum_{\varsigma=1}^{q_2} \mu_\varsigma \frac{\Gamma(\delta_2)}{\Gamma(\delta_2+\alpha_\varsigma)} (\ln \varrho_2)^{\delta_2+\alpha_\varsigma-1} - b_2 \sum_{\tau=1}^{p_2} \pi_\tau (\ln \eta_\tau)^{q_2-1} = \frac{1}{8}\sqrt{\pi} > 0.$$

Thus condition (F1) is satisfied. Choose

$$a_0(x) = \frac{2x}{(9+x)^2},\ a_1(x) = \frac{xe^{-x}}{[1+(\ln x)^{1.5}+(\ln x)^{1.5}]^{0.1}},\ a_2(x) = \frac{xe^{-2x}}{[1+(\ln x)^{0.5}+(\ln x)^{0.1}]^{0.3}},$$
$$a_3(x) = xe^{-10x},\ a_4(x) = \frac{x}{1+x^2},$$
$$b_0(x) = \frac{x}{20(1+x^2)},\ b_1(x) = \frac{xe^{-3x}}{[1+(\ln x)^{1.5}+(\ln x)^{1.5}]^{0.2}},\ b_2(x) = \frac{xe^{-4x}}{[1+(\ln x)^{1.5}+(\ln x)^{1.5}]^{0.4}},$$
$$b_3(x) = \frac{3x^3}{(3+x^3)^2},\ b_4(x) = \frac{s}{10(1+x^2)}.$$

Thus we get

$$|H_{1(u,v)}| \le a_0(x) + a_1(x)|u|^{0.1} + a_2(x)|v|^{0.3} + a_3(x)|l|^{0.4} + a_4(x)|m|^{0.1},$$
$$|H_{2(u,v)}| \le b_0(x) + b_1(x)|u|^{0.2} + b_2(x)|v|^{0.4} + b_3(x)|l|^{0.2} + b_3(x)|m|^{0.3},$$

where $\theta_1 = 0.1, \theta_2 = 0.3, \theta_3 = 0.4, \theta_4 = 0.1, \vartheta_1 = 0.2, \vartheta_2 = 0.4, \vartheta_3 = 0.2, \vartheta_4 = 0.3$, and

$$a_0^* = \int_1^{+\infty} a_0(x)\frac{\mathrm{d}x}{x} = 0.200000,\quad b_0^* = \int_1^{+\infty} b_0(x)\frac{\mathrm{d}x}{x} = 0.039200,$$
$$a_1^* = \int_1^{+\infty} a_1(x)[1+(\ln x)^{0.5}+(\ln x)^{0.1}]^{0.1}\frac{\mathrm{d}x}{x} = 0.367879,$$
$$a_2^* = \int_1^{+\infty} a_2(x)[1+(\ln x)^{0.5}+(\ln x)^{0.1}]^{0.3}\frac{\mathrm{d}x}{x} = 0.183940,$$
$$a_3^* = \int_1^{+\infty} a_3(x)\frac{\mathrm{d}x}{x} = 0.036788,\quad a_4^* = \int_1^{+\infty} a_4(x)\frac{\mathrm{d}x}{x} = 0.785398,$$
$$b_1^* = \int_1^{+\infty} b_1(x)[1+(\ln s)^{0.5}+(\ln s)^{0.1}]^{0.2}\frac{\mathrm{d}x}{x} = 0.122626,$$
$$b_2^* = \int_1^{+\infty} b_2(x)[1+(\ln s)^{0.5}+(\ln s)^{0.1}]^{0.4}\frac{\mathrm{d}x}{x} = 0.091970,$$
$$b_3^* = \int_1^{+\infty} b_3(x)\frac{\mathrm{d}x}{x} = 0.250000,\quad b_4^* = \int_1^{+\infty} b_4(x)\frac{\mathrm{d}x}{x} = 0.078540.$$

Therefore condition (F2) is satisfied.

It is easy to verify that the functions e_1, e_2 are monotonically increasing in the variables u, v, l, m. Moreover, for all $x \in J$, it holds that $e_1(x,0,0,0,0) \not\equiv 0$ and $e_2(x,0,0,0,0) \not\equiv 0$. Hence, condition (F3) is satisfied. By Theorem 3.1, we conclude that the HFDS (3.9) admits two sets of positive solutions, which can be constructed by means of the two families of monotone iterative sequences given in (3.1) and (3.2).

4. Conclusion

We propose a novel partial order defined on the product space and employ the monotone iterative technique to examine a HFDS over the half-line. This system integrates high-order fractional derivatives and incorporates more general discrete boundary value conditions. Our approach aims to construct two distinct families of explicit monotone iterative sequences, which converge to the minimal and maximal positive solutions respectively.

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