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STATIONARY DISTRIBUTION AND PERSISTENCE OF A STOCHASTIC PREDATOR-PREY MODEL WITH A FUNCTIONAL RESPONSE*

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Abstract A stochastic predator-prey model with a functional response is investigated in this paper. The asymptotic properties of the stochastic model are considered here. Under some conditions, we show that the stochastic model is persistent in mean. Moreover, the existence of stationary distribution to the model is obtained. Simulations are also carried out to confirm our analytical results.

Keywords Predator-prey, stationary distribution, persistence in mean.

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1. Introduction

Dynamic relationship between predators and their preys has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [4]. And predator′s functional response is a important factor of predator-prey relationship that stands for the consumption rate of each predator on prey. In the predator-prey model, Crowley and Martin [2] proposed the functional response: $p(x,y) = \frac{fx(t)y(t)}{(1+\alpha_1 x(t)+\alpha_2 y(t)+\alpha_3 x(t)y(t))}$, which is an improvement of Holling types and Beddington-DeAngelis functional response. When $\alpha_1 = \alpha_2 = \alpha_3 = 0$, the Crowley-Martin type of functional response is simplified to a linear mass-action function response (or Holling type I functional response), when $\alpha_2 = \alpha_3 = 0$, the functional response is Holling type II functional response, and it is Beddington-DeAngelis functional response if $\alpha_3 = 0$.

As the item $1+\alpha_1 x(t)+\alpha_2 y(t)+\alpha_3 x(t)y(t)$ can be changed to $\Big(1+\alpha x(t)\Big)\Big(1+\beta y(t)\Big)$, the predator-prey model takes the form

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t)\left(a - x(t) - \frac{by(t)}{(1+\alpha x(t))(1+\beta y(t))}\right), \\ \frac{dy(t)}{dt} &= y(t)\left(c - y(t) + \frac{fx(t)}{(1+\alpha x(t))(1+\beta y(t))}\right), \end{aligned} \tag{1.1}$$

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where x, y represent the densities of prey and predator respectively, the parameters a, b, c, f are positive constants and α, β are non-negative constants; c may change sign and $c > 0$ indicates the predator has other resource. Many deterministic predator-prey models with the functional response have been investigated extensively by the scholars [9, 21, 22, 24–27] and the references therein.

Because population systems are always subject to environmental noises, it is therefore necessary to find out how the noises affect population systems. Recently, asymptotic properties of stochastic population models have been extensively considered in the literature [3, 7, 10–18, 20, 23, 28, 29], among others.

It has been pointed out that the growth rates of species are often subject to white noise [1]. The growth rates are estimated by average values plus error terms [30], and the error terms follow normal distributions. Taking into account the effect of randomly fluctuating environment, we incorporate white noise in each equation of the system (1.1). The stochastic system has the form

$$\begin{aligned} dx(t) &= x(t)\left(a - x(t) - \frac{by(t)}{(1+\alpha x(t))(1+\beta y(t))}\right)dt + \sigma_1 x(t)dB_1(t), \\ dy(t) &= y(t)\left(c - y(t) + \frac{fx(t)}{(1+\alpha x(t))(1+\beta y(t))}\right)dt + \sigma_2 y(t)dB_2(t), \end{aligned} \tag{1.2}$$

where $B_i(t), i = 1, 2$ are independent standard Brownian motions.

The global existence, uniqueness, boundedness of positive solution and stochastic permanence (see [8]) were obtained for the stochastic model (1.2) [31]. For the stochastic population model, the following questions are also interesting:

i) Generally speaking, when the perturbation is large, the population will be forced to expire whilst it remains persistent when the perturbation is small. Naturally, under which conditions, is the stochastic system persistent in mean [7] ?

ii) The existence of an absorbing state is mathematically interesting for deterministic models. It is natural to take a clue from the deterministic models and study the counterpart of the stochastic models. What happens if the noise is not strong? Is there a stationary distribution of the stochastic model (1.2)?

To the best of our knowledge, there is no work has been done on the above questions to the stochastic system (1.2). Motivated by the above discussions, we aim to study the above properties of the stochastic model (1.2). In this paper, we obtain the stochastic system is persistent in mean. Moreover, we obtain the sufficient conditions for the existence of stationary distribution to the stochastic model. Finally we introduce some figures to illustrate the main results.

2. Asymptotic properties

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets.) Consider a stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t)$$

on $t \geq 0$ with initial value $x(0) = x_0 \in R^n$, where $f : R^n \times R_+ \to R^n$ and $g : R^n \times R_+ \to R^{n\times m}$.

Lemma 2.1 (see [19]). *Let $p \geq 2$ and $X_0 = (x_0, y_0) \in R^2_+$. Assume that there exists a constant $\alpha > 0$ such that for all $(X,t) \in R^2_+ \times [0,T]$,*

$$X^T f(X,t) + \frac{p-1}{2}|g(X,t)|^2 \leq \alpha\Big(1 + |X|^2\Big).$$

Then

$$E|X(t)|^p \leq 2^{\frac{p-2}{2}}\Big(1 + E|X_0|^p\Big)e^{p\alpha t} \quad for\ all \quad t \in [0,T].$$

Theorem 2.1. *For any initial value $X_0 = (x_0, y_0) \in R^2_+$, the solution of (1.2) satisfies*

$$E|X(t)|^p \leq 2^{\frac{p-2}{2}}\Big(1 + E|X_0|^p\Big)e^{p\alpha t} \quad for\ all \quad t \in [0,T].$$

Proof. For the model (1.2), we have

$$\begin{aligned}
&X^T f(X,t) + \tfrac{p-1}{2}|g(X,t)|^2 \\
&= ax^2 - x^3 - \tfrac{bx^2y}{(1+\alpha x)(1+\beta y)} + cy^2 - y^3 + \tfrac{fxy^2}{(1+\alpha x)(1+\beta y)} + \tfrac{p-1}{2}\Big(\sigma_1^2 x^2 + \sigma_2^2 y^2\Big) \\
&\leq \Big(a + \tfrac{p-1}{2}\sigma_1^2\Big)x^2 + \tfrac{f}{\alpha\beta}y + \Big(c + \tfrac{p-1}{2}\sigma_2^2\Big)y^2.
\end{aligned}$$

Obviously there exist $K_2 > 0$ and $K_3 > 0$ such that

$$\frac{f}{\alpha\beta}y + \Big(c + \frac{p-1}{2}\sigma_2^2\Big)y^2 \leq K_2 y^2 + K_3.$$

Let $K_1 = \max\Big\{a + \frac{p-1}{2}\sigma_1^2, K_2, K_3\Big\}$, then

$$\Big(a + \frac{p-1}{2}\sigma_1^2\Big)x^2 + \frac{f}{\alpha\beta}y + \Big(c + \frac{p-1}{2}\sigma_2^2\Big)y^2 \leq K_1x^2 + K_1y^2 + K_1 = K_1\big(1 + |X|^2\big).$$

Thus for $p \geq 2$, we get

$$E|X(t)|^p \leq 2^{\frac{p-2}{2}}\Big(1 + E|X(t_0)|^p\Big)e^{p\alpha t},$$

for all $t \in [0,T]$. □

Lemma 2.2 (see [19]). *Suppose there exists a pair of positive constants γ and ρ such that*

$$X^T f(X,t) \leq \gamma|X|^2 + \rho \quad for\ all \quad (X,t) \in R^2_+ \times [0,\infty].$$

Then the solution has the property

$$\lim_{t\to\infty} \frac{|X(t)|}{e^{\gamma t}\sqrt{\ln\ln t}} = 0 \quad a.s.$$

Theorem 2.2. *The solution of (1.2) satisfies*

$$\lim_{t\to\infty} \frac{|X(t)|}{e^{\gamma t}\sqrt{\ln\ln t}} = 0 \quad a.s.$$

Proof. It follows from (1.2) that

$$\begin{aligned} X^T f(X,t) &= ax^2 - x^3 - \tfrac{bx^2y}{(1+\alpha x)(1+\beta y)} + cy^2 - y^3 + \tfrac{fxy^2}{(1+\alpha x)(1+\beta y)} \\ &\le ax^2 + cy^2 + \tfrac{f}{\alpha\beta} y. \end{aligned}$$

Obviously when $K_2(K_1 - c) \ge \frac{f^2}{4\alpha^2\beta^2},\ K_1 \ge c,\ K_2 \ge 0$, we have

$$cy^2 + \frac{f}{\alpha\beta} y \le K_1 y^2 + K_2 \quad (y \ge 0).$$

Let $\gamma = \max\{a, K_1\},\ \rho = K_2$, then

$$X^T f(X,t) \le \gamma |X|^2 + \rho.$$

Therefore

$$\lim_{t\to\infty} \frac{|X(t)|}{e^{\gamma t}\sqrt{\ln\ln t}} = 0 \quad \text{for all} \quad (X,t) \in R^d_+ \times [0,\infty].$$

□

Theorem 2.3. *The solution of (1.2) has the property*

$$\limsup_{t\to\infty} \frac{\ln\big(x(t)y(t)\big)}{\ln t} \le 2 \quad a.s.$$

Proof. Define the function $V = e^{rt}\ln x$ for $(x,y) \in R^2_+$ and $r > 0$. By the *Itô* formula we obtain

$$dV = re^{rt}\ln x(t)dt + \left(a - x(t) - \frac{by(t)}{(1+\alpha x(t))(1+\beta y(t))}\right)e^{rt}dt - \frac{\sigma_1^2}{2}e^{rt}dt + \sigma_1 e^{rt}dB_1(t).$$

Hence

$$e^{rt}\ln x(t) = \ln x(0) + \int_0^t \Big[r\ln x(s) + a - x(s) - \frac{by(s)}{(1+\alpha x(s))(1+\beta y(s))} - \frac{\sigma_1^2}{2}\Big]e^{rs}ds + M_1(t),$$

where $M_1(t) = \displaystyle\int_0^t e^{rs}\sigma_1 dB_1(s)$ is a real-valued continuous local martingale.

For all $\varepsilon \in (0,1),\ \theta > 1,\ k \ge 1$, by exponential martingale inequality we have

$$P\Big\{\sup_{0\le t\le k}\Big[M_1(t) - \tfrac{\varepsilon}{2}e^{-rk}\Big\langle M_1(t), M_1(t)\Big\rangle\Big] > \tfrac{\theta e^{rk}}{\varepsilon}\ln k\Big\} \le \tfrac{1}{k^\theta}.$$

Using Borel-Cantelli's lemma, we can find $\Omega_1 \subset \Omega$ and $P(\Omega_1) = 1$, there exists $k_1(\omega)$ such that

$$M_1(t) \le \frac{\varepsilon}{2}e^{-rk}\Big\langle M_1(t), M_1(t)\Big\rangle + \frac{\theta e^{rk}}{\varepsilon}\ln k,$$

for $\omega \in \Omega_1$ and $0 \le t \le k/2,\ k \ge k_1(\omega)$. Thus

$$\begin{aligned} e^{rt}\ln x(t) \le \ln x(0) &+ \int_0^t e^{rs}\Big[r\ln x(s) + a - x(s) - \frac{by(s)}{(1+\alpha x(s))(1+\beta y(s))} - \frac{\sigma_1^2}{2}\Big]ds \\ &+ \tfrac{\varepsilon}{2}e^{-rk}\int_0^t e^{2rs}\sigma_1^2 ds + \frac{\theta e^{rk}}{\varepsilon}\ln k, \end{aligned}$$

for $\omega \in \Omega_1$ and $0 \le t \le k/2,\ k \ge k_1(\omega)$.

Similarly, there exist $\Omega_2 \subset \Omega$, $P(\Omega_2) = 1$ and $k_2(\omega)$, such that

$$\begin{aligned} e^{rt} \ln y(t) \le \ln y(0) + \int_0^t e^{rs}\Big[& r\ln y(s) + c - y(s) + \frac{fx(s)}{(1+\alpha x(s))(1+\beta y(s))} \\ & - \tfrac{(1-\varepsilon e^{-r(k-2s)})\sigma_2^2}{2}\Big] ds + \tfrac{\theta}{\varepsilon} e^{rk} \ln k, \end{aligned}$$

for $\omega \in \Omega_2$ and $0 \le t \le k/2,\ k \ge k_2(\omega)$. Let $\Omega_0 = \Omega_1 \bigcap \Omega_2$, obviously $P(\Omega_0) = 1$. For any $\omega \in \Omega_0$, let $k_0(\omega) = \max\Big\{k_1(\omega), k_2(\omega)\Big\}$, then

$$\begin{aligned} & e^{rt} \ln\big(x(t)y(t)\big) \\ \le\ & \ln\big(x(0)y(0)\big) + \tfrac{2\theta}{\varepsilon} e^{rk}\ln k + \int_0^t e^{rs}\Big[a - x(s) - \frac{by(s)}{(1+\alpha x(s))(1+\beta y(s))} + c - y(s) \\ & - \tfrac{1-\varepsilon e^{-r(k-2s)}}{2}\sigma_1^2 + \tfrac{fx(s)}{(1+\alpha x(s))(1+\beta y(s))} - \tfrac{1-\varepsilon e^{-r(k-2s)}}{2}\sigma_2^2 + r\ln\big(x(s)y(s)\big)\Big] ds, \end{aligned}$$

for $\omega \in \Omega_0$ and $0 \le t \le k/2,\ k \ge k_0(\omega)$. Obviously there exists $M > 0$ such that

$$a - x - \frac{1-\varepsilon e^{-r(k-2s)}}{2}\big(\sigma_1^2 + \sigma_2^2\big) + c - y + \frac{fx - by}{(1+\alpha x)(1+\beta y)} + r\ln(xy) \le M,$$

for $0 \le s \le t \le k/2$, $(x, y) \in R_+^2$. Therefore

$$\ln\big(x(t)y(t)\big) \le \frac{\ln\big(x(0)y(0)\big)}{e^{rt}} + \frac{2\theta}{\varepsilon}\ln k + \frac{M}{r}.$$

For $k \le t \le k+1$, now let $k \to \infty$ and we obtain

$$\frac{\ln\big(x(t)y(t)\big)}{\ln t} \le \frac{2\theta}{\varepsilon}.$$

We know that $\varepsilon \in (0, 1)$ and $\theta > 1$, so we can let $\varepsilon \to 1,\ \theta \to 1$, then

$$\limsup_{t\to\infty} \frac{\ln\big(x(t)y(t)\big)}{\ln t} \le 2 \quad a.s.$$

The proof is complete. □

3. Persistence in mean

Firstly, we introduce a fundamental lemma which will be used.

Lemma 3.1 (see [7]). *Consider one-dimensional stochastic differential equation*

$$dx = x[a - bx]dt + \sigma x dB(t), \tag{3.1}$$

where a, b, σ are positive, and $B(t)$ is standard Brownian motion. Under the condition $a > \frac{\sigma^2}{2}$, for any initial value $x_0 > 0$, the solution $x(t)$ to (3.1) has the properties

$$\lim_{t\to\infty} \frac{\ln x(t)}{t} = 0 \quad a.s.$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds = \frac{a - \frac{\sigma^2}{2}}{b} \quad a.s.$$

To continue our analysis, let us impose hypotheses as follows.

Assumption 1: $a - \frac{b}{\beta} - \frac{\sigma_1^2}{2} > 0.$

Assumption 2: $c - \frac{\sigma_2^2}{2} > 0.$

We consider the following results by comparison theorem. It is obvious that

$$dx(t) \leq x(t)\big[a - x(t)\big]dt + \sigma_1 x(t)dB_1(t),$$

and

$$dy(t) \leq y(t)\Big[c + \frac{f}{\alpha} - y(t)\Big]dt + \sigma_2 y(t)dB_2(t).$$

Denote that $X_2(t)$ is the solution to the following stochastic equation

$$dX_2(t) = X_2(t)\big[a - X_2(t)\big]dt + \sigma_1 X_2(t)dB_1(t),\ X_2(0) = x(0), \tag{3.2}$$

and $Y_2(t)$ is the solution of the equation

$$dY_2(t) = Y_2(t)\Big[c + \frac{f}{\alpha} - Y_2(t)\Big]dt + \sigma_2 Y_2(t)dB_2(t),\ Y_2(0) = y(0). \tag{3.3}$$

We have

$$x(t) \leq X_2(t),\ y(t) \leq Y_2(t),\ t \in [0, +\infty) \quad a.s.$$

Moreover, we can get

$$dx(t) \geq x(t)\Big[a - \frac{b}{\beta} - x(t)\Big]dt + \sigma_1 x(t)dB_1(t),$$

and

$$dy(t) \geq y(t)\big[c - y(t)\big]dt + \sigma_2 y(t)dB_2(t).$$

Similarly, we denote $X_1(t)$ is the solution of stochastic equation

$$dX_1(t) = X_1(t)\Big[a - \frac{b}{\beta} - X_1(t)\Big]dt + \sigma_1 X_1(t)dB_1(t),\ X_1(0) = x(0), \tag{3.4}$$

and the stochastic system

$$dY_1(t) = Y_1(t)\big[c - Y_1(t)\big]dt + \sigma_2 Y_1(t)dB_2(t),\ Y_1(0) = y(0), \tag{3.5}$$

has the unique solution $Y_1(t)$. Consequently

$$x(t) \geq X_1(t),\ y(t) \geq Y_1(t),\ t \in [0, +\infty) \quad a.s.$$

To sum up, we have

$$X_1(t) \leq x(t) \leq X_2(t),\ Y_1(t) \leq y(t) \leq Y_2(t),\ t \in [0, +\infty) \quad a.s. \tag{3.6}$$

Lemma 3.2. *Under Assumption 1 and Assumption 2, for any initial value* $X_0 = (x_0, y_0) \in R_+^2$*, the solution* $X(t) = \big(x(t), y(t)\big)$ *to (1.2) satisfies*

$$\lim_{t\to\infty} \frac{\ln x(t)}{t} = 0,\ \lim_{t\to\infty} \frac{\ln y(t)}{t} = 0 \quad a.s.$$

Proof. Lemma 3.1 and inequalities (3.6) can straightforward deduce the desired assertion. □

Lemma 3.3. *Suppose Assumption 1 and Assumption 2. Then for any initial value* $X_0 = (x_0, y_0) \in R_+^2$, $X_1(t)$, $X_2(t)$, $Y_1(t)$, $Y_2(t)$ *obey*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_1(s)ds = a - \frac{b}{\beta} - \frac{\sigma_1^2}{2}, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t X_2(s)ds = a - \frac{\sigma_1^2}{2} \quad a.s.$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Y_1(s)ds = c - \frac{\sigma_2^2}{2}, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t Y_2(s)ds = c + \frac{f}{\alpha} - \frac{\sigma_2^2}{2} \quad a.s.$$

Proof. Lemma 3.1 and equations (3.2)-(3.5) can imply the conclusion. □

Theorem 3.1. *Under Assumption 1 and Assumption 2, the stochastic model (1.2) is persistent in mean.*

Proof. By virtue of Lemma 3.3 and inequalities (3.6), we obtain

$$0 < a - \frac{b}{\beta} - \frac{\sigma_1^2}{2} \le \lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \le a - \frac{\sigma_1^2}{2} \quad a.s.$$

and

$$0 < c - \frac{\sigma_2^2}{2} \le \lim_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds \le c + \frac{f}{\alpha} - \frac{\sigma_2^2}{2} \quad a.s.$$

Therefore the stochastic system is persistent in mean. □

4. Stationary distribution

Lemma 4.1 (see [5]). *There exists a bounded domain $U \subset E_n$ with regular boundary Γ, having the properties that:*

(A1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(A2) If $x \in E_n \setminus U$,the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < +\infty$ for every $K \subset E_n$ that is a compact subset.

Then the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$ with density in E_n such that for any Borel set $B \subset E_n$, $\lim_{t\to\infty} P(t, x, B) = \mu(B)$, and

$$P\Big\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(x(s))ds = \int_{E_n} f(x)\mu(dx)\Big\} = 1.$$

To validate assumptions (A1) and (A2), it suffices to prove that there exist some neighborhood U and a nonnegative C^2-function such that $A(x)$ is uniformly elliptical in U $\Big($i.e. $A(x)$ is symmetric and satisfies $\kappa_1|\xi|^2 \le \big\langle A(x)\xi,\ \xi\big\rangle \le \kappa_1^{-1}|\xi|^2$ for all $\xi \in U$ and some constant $\kappa_1 \in (0, 1]\Big)$and for any $x \in E_n \setminus U$, $LV(x) \le c$ for some $c < 0$ (for details refer to [32]).

Theorem 4.1. *Assume that Assumption 1 and Assumption 2 hold. Then the s-tochastic model (1.2) has a stationary distribution and has the ergodic property:*

$$P\Big\{\lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)ds = \int_{R_+} z\mu(dz)\Big\} = 1.$$

Proof. Denote $V(X,t) = x + \frac{1}{x^\gamma} + y + \frac{1}{y^\gamma}$ for $\gamma > 0$, in view of *Itô* formula, we compute

$$\begin{aligned}
LV = {} & ax - x^2 - \tfrac{bxy}{(1+\alpha x)(1+\beta y)} - \tfrac{\gamma a}{x^\gamma} + \tfrac{\gamma}{x^{\gamma-1}} + \tfrac{\gamma by}{x^\gamma(1+\alpha x)(1+\beta y)} \\
& + cy - y^2 + \tfrac{fxy}{(1+\alpha x)(1+\beta y)} - \tfrac{\gamma c}{y^\gamma} + \tfrac{\gamma}{x^{\gamma-1}} - \tfrac{\gamma dy}{y^\gamma(1+\alpha x)(1+\beta y)} \\
& + \tfrac{\gamma(\gamma+1)\sigma_1^2}{2x^\gamma} + \tfrac{\gamma(\gamma+1)\sigma_2^2}{2y^\gamma} \\
\le {} & ax - x^2 + \tfrac{\gamma by}{x^\gamma(1+\alpha x)(1+\beta y)} - \tfrac{\gamma a}{x^\gamma} + \tfrac{\gamma}{x^{\gamma-1}} + cy - y^2 \\
& + \tfrac{fxy}{(1+\alpha x)(1+\beta y)} - \tfrac{\gamma c}{y^\gamma} + \tfrac{\gamma}{y^{\gamma-1}} + \tfrac{\gamma(\gamma+1)\sigma_1^2}{2x^\gamma} + \tfrac{\gamma(\gamma+1)\sigma_2^2}{2y^\gamma} \\
\le {} & ax - x^2 + \tfrac{\gamma b}{\beta x^\gamma} - \tfrac{\gamma a}{x^\gamma} + \tfrac{\gamma}{x^{\gamma-1}} + cy - y^2 + \tfrac{f}{\alpha\beta} - \tfrac{\gamma c}{y^\gamma} \\
& + \tfrac{\gamma}{y^{\gamma-1}} + \tfrac{\gamma(\gamma+1)\sigma_1^2}{2x^\gamma} + \tfrac{\gamma(\gamma+1)\sigma_2^2}{2y^\gamma} \\
= {} & ax - x^2 + \gamma x^{1-\gamma} - \gamma\tfrac{a - \frac{b}{\beta} - \frac{1}{2}\gamma(\gamma+1)\sigma_1^2}{x^\gamma} + cy - y^2 \\
& + \gamma y^{1-\gamma} - \gamma\tfrac{c - \frac{1}{2}(\gamma+1)\sigma_2^2}{y^\gamma} + \tfrac{f}{\alpha\beta}.
\end{aligned}$$

Under Assumption 1 and Assumption 2, obviously when $\gamma \to 0+$, there exist two positive constants N and M, such that $LV \le -M$ in $R_+^2 \setminus \Phi$, where $\Phi = \Big\{(x,y) \in R_+^2 \big| \frac{1}{N} \le x \le N,\ \frac{1}{N} \le y \le N\Big\}$.

The equations (1.2) can be written as:

$$d\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\Big(a - x - \frac{by}{(1+\alpha x)(1+\beta y)}\Big) \\ y\Big(c - y + \frac{fx}{(1+\alpha x)(1+\beta y)}\Big) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 x_1 \\ 0 \end{pmatrix} dB_1(t) + \begin{pmatrix} 0 \\ \sigma_2 y \end{pmatrix} dB_2(t).$$

The diffusion matrix is

$$A(x,y) = \begin{pmatrix} \sigma_1^2 x^2 & 0 \\ 0 & \sigma_2^2 y^2 \end{pmatrix}.$$

For $(x,y) \in \Phi,\ \xi = \big(\xi_1, \xi_2\big) \in R^2$, we have

$$\sum_{i,j=1}^{2} a_{ij}(x,y)\xi_i\xi_j = \sigma_1^2 x^2 \xi_1^2 + \sigma_2^2 y^2 \xi_2^2.$$

Then we can find $M_1 > 0$ such that $\min\big\{\sigma_1^2 x^2, \sigma_2^2 y^2\big\} \ge M_1$. Hence for $(x,y) \in \Phi,\ \xi = (\xi_1, \xi_2) \in R^2$, we obtain

$$\sum_{i,j=1}^{2} a_{ij}(x,y)\xi_i\xi_j = \sigma_1^2 x^2 \xi_1^2 + \sigma_2^2 y^2 \xi_2^2 \ge M_1|\xi|^2.$$

Therefore the system has a stationary distribution and has the ergodic property:

$$P\bigg\{\lim_{T\to\infty} \frac{1}{T}\int_0^T x(s)ds = \int_{R_+} z\mu(dz)\bigg\} = 1.$$

$\square$

5. Numerical simulations

In this section we will use the Milstein method [6] to illustrate the analytical results.

Consider the discretization equation:

$$x_{k+1} = x_k + x_k\left[a - x_k - \frac{by_k}{(1+\alpha x_k)(1+\beta y_k)}\right]\Delta t + \sigma_1 x_k\sqrt{\Delta t}\xi_k + \frac{\sigma_1^2}{2}x_k(\xi_k^2 - 1)\Delta t,$$

$$y_{k+1} = y_k + y_k\left[c - y_k + \frac{fx_k}{(1+\alpha x_k)(1+\beta y_k)}\right]\Delta t + \sigma_2 y_k\sqrt{\Delta t}\eta_k + \frac{\sigma_2^2}{2}y_k(\eta_k^2 - 1)\Delta t,$$

where ξ_k and η_k are Gaussian random variables that follow $N(0,1)$.

In all figures, we choose $a = 0.6, c = 0.3, b = 0.1, f = 0.35, \alpha = 0.4, \beta = 1$ and $\sigma_1 = \sigma_2 = 0.1$. Obviously, Assumption 1 and Assumption 2 are all satisfied.

Under Assumption 1 and Assumption 2, we consider its asymptotic behaviors of the stochastic model, see Figure 1.

In Figure 2 and Figure 3, we choose $x_0 = 1.0$ and $y_0 = 0.5$ respectively. The stochastic system is persistent in mean. Figure 2 and Figure 3 confirm these.

We choose $x_0 = 0.44$, $y_0 = 1.6$ in Figure 4. Then by virtue of Theorem 4.1, system (1.2) has a stationary distribution. See Figure 4.

6. Conclusions

Owing to its theoretical and practical significance, predator-prey system has deserved a lot of attention. One significant component of the predator-prey relationship is the functional responses, such as, Holling types I-III, Hassell-Varley type, Beddington-DeAngelis type and so on. Crowley and Martin [2] proposed a new type of response function, which is an improvement of Holling type II and Beddington-DeAngelis functional response. Many authors have investigated deterministic predator-prey systems with the functional response. In reality, most natural phenomena do not obey strictly deterministic laws but rather oscillate randomly. Motivated by it, we aim to investigate the asymptotic properties of the stochastic model. Our work is the first attempt to consider its stationary distribution of the stochastic predator-prey system with the functional response. Here, since population system is perturbed by environmental noises, we devoted to revealing the relationships between the coefficients of population model and the intensities of environmental noises.

Considering under what conditions interacting populations, coexistence is a basic topic of theoretical and practical importance in population dynamics. For the deterministic model (1.1), under some conditions, the stability of positive equilibrium and coexistence are investigated. To better understand the interactions between deterministic and stochastic forces, many existing results aim to investigate that environmental noises promote or weaken species coexistence. For stochastic model (1.2), we established the sufficient conditions for persistence in mean. Moreover, stationary distribution reflects stability of stochastic population systems. Under conditions $a - \frac{b}{\beta} - \frac{\sigma_1^2}{2} > 0$ and $c - \frac{\sigma_2^2}{2} > 0$, the existence of stationary distribution to the model is obtained. To a great extent, provided that the random perturbations are sufficiently small, some nice properties such as persistence and stationary distribution are retained.

Our results have certain implications in theoretically. Although population systems are always subject to environmental noises, under conditions $a - \frac{b}{\beta} - \frac{\sigma_1^2}{2} > 0$ and $c - \frac{\sigma_2^2}{2} > 0$, we can assume that there is a positive environmental. This may be because small environmental perturbations may induce small death rates and large birth rates.

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