EXACT SOLUTIONS AND DYNAMICS OF THE RAMAN SOLITON MODEL IN NANOSCALE OPTICAL WAVEGUIDES, WITH METAMATERIALS, HAVING PARABOLIC LAW NON-LINEARITY*

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Abstract This paper investigate the Raman soliton model in nanoscale optical waveguides, with metamaterials, having parabolic law non-linearity by using the method of dynamical systems. The functions $q(x,t) = \phi(\xi) \exp(i(-kx+\omega t))$ are solutions of the equation (1.1) that governs the propagation of Raman solitons through optical metamaterials, where $\xi = x - vt$ and $\phi(\xi)$ in the solutions satisfy a singular planar dynamical system (1.5) which has two singular straight lines. By using the bifurcation theory method of dynamical systems to the equation of $\phi(\xi)$, bifurcations of phase portraits for this dynamical system are obtained under 28 different parameter conditions. Based on those phase portraits, 62 exact solutions of system (1.5) including periodic solutions, heteroclinic and homoclinic solutions, periodic peakons and peakons as well as compacton solutions are derived.

Keywords Integrable system, exact solution, homoclinic and heteroclinic orbit, periodic solution, peakon, compacton.

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1. Introduction

Recently, Xu, et al. [11] stated that the dynamics of temporal optical solitons is a treasuretrove in the area of non-linear optics. The starting point is Maxwell's equation from electromagnetic theory. Electromagnetic properties of complex materials, with simultaneous negative dielectric permittivity and magnetic permeability, also known as double negative material, have attracted a lot of attention in research in the last decades. Novel and interesting features of these engineered materials, that are also known as metamaterials, and their possible applications to support short duration optical soliton pulses have been investigated.

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The dimensionless form of non-linear Schröinger's equation (NLSE) that governs the propagation of Raman solitons through optical metamaterials, is given by (see [11] and [10])

$$iq_t + aq_{xx} + (c_1|q|^2 + c_2|q|^4 + c_3|q|^6)q$$

= $i\alpha q_x + i\lambda(|q|^2q)_x + i\nu(|q|^2)_x q + \theta_1(|q|^2q)_{xx} + \theta_2|q|^2q_{xx} + \theta_3q^2q_{xx}^*,$ (1.1)

where $a \neq 0$ and q(x,t) represents the complex-valued wave function with the independent variables being x and t that represent spatial and temporal variables, respectively. On the right-hand side of (1.1), α represents the coefficient of intermodal dispersion. This arises when the group velocity of light propagating through a metamaterial is dependent on the propagation mode in addition to chromatic dispersion. The factors λ and ν are accounted for self-steepening for preventing shock-waves, and non-linear dispersion. The terms with θ_j , j = 1, 2, 3 arise in the context of optical metamaterials. We notice that c_j , j = 1, 2, 3 are coefficients of the non-linear terms and together form the polynomial-law nonlinearity.

We have studied the model (1.1) (see [14]) with $c_2 = c_3 = 0$ and $c_1 \neq 0$ which collapses to the kerr-law nonlinearity, and obtained some interesting results. Now, we make $c_3 = 0$ and $c_1 \neq 0, c_2 \neq 0$ in model (1.1), and obtain the equation

$$iq_t + aq_{xx} + (c_1|q|^2 + c_2|q|^4)q$$

$$= i\alpha q_x + i\lambda (|q|^2q)_x + i\nu (|q|^2)_x q + \theta_1 (|q|^2q)_{xx} + \theta_2 |q|^2 q_{xx} + \theta_3 q^2 q_{xx}^*.$$
(1.2)

The equation (1.2) is explained as the parabolic-law nonlinearity. Consider the solutions of equation (1.2) having the form

$$q(x,t) = \phi(\xi) \exp(i(-kx + \omega t)), \quad \xi = x - vt,$$
 (1.3)

where $\phi(\xi)$ represents the wave profile, k and ω represent the soliton frequency and wave number respectively, v is speed of the wave. Substituting (1.3) into (1.2) and separating the real and imaginary parts, we have

$$[a - (3\theta_1 + \theta_2 + \theta_3)\phi^2]\phi'' - a_1\phi + a_3\phi^3 + c_2\phi^5 - 6\theta_1\phi(\phi')^2 = 0, \qquad (1.4_r)$$

$$(v + \alpha + 2ak) + [3\lambda + 2\nu - 2k(3\theta_1 + \theta_2 - \theta_3)]\phi^2 = 0, \qquad (1.4i)$$

where $a_1 = \omega + \alpha k + ak^2$, $a_3 = c_1 - k\lambda + (\theta_1 + \theta_2 + \theta_3)k^2$, the notation $\phi' = \frac{d\phi}{d\xi}$. The imaginary part equation (1.4_i) , upon setting the coefficients of linearly independent functions to zero, gives the relations:

$$v = -(\alpha + 2ak), \quad 3\lambda + 2\nu = 2k(3\theta_1 + \theta_2 - \theta_3).$$

Assume that $\theta_1 \neq 0$ and $3\theta_1 + \theta_2 + \theta_3 \neq 0$. Making the parameter transformation: $a_0 = \frac{a}{3\theta_1 + \theta_2 + \theta_3}, \beta = \frac{6\theta_1}{3\theta_1 + \theta_2 + \theta_3}, \alpha_0 = \frac{a_1}{6\theta_1}, \alpha_2 = -\frac{a_3}{6\theta_1}, \alpha_4 = -\frac{c_2}{6\theta_1}$, equation (1.4_r) is equivalent to the following planar dynamical system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\beta\phi(y^2 + \alpha_0 + \alpha_2\phi^2 + \alpha_4\phi^4)}{a_0 - \phi^2}.$$
 (1.5)

System (1.5) is a five-parameter integrable planar dynamical system depending on the parameter group $(a_0, \beta, \alpha_0, \alpha_2, \alpha_4)$.

We notice that in [11], the author considered the case of $\theta_j = 0, j = 1, 2, 3$, in the equation (1.1) and obtained an implicit solution. Recently, Biswas, et al. [1,2] considered the case of $c_2 = 0$ for equation (1.1). By using the so called "the simplest equation method", they obtained a few exact solutions of system (1.5) with $\alpha_4 = 0$.

More recently, Zhou, et al. [12, 13] and Sonmezoglu et al. [9] used the extended trial equation method to get some exact travelling wave solutions of equation (1.1). Unfortunately, the dynamical behavior of the traveling wave system (1.5) depending on the parameters has not been studied completely before. Recently, in paper [14], we have used the method of dynamical system to investigate the dynamical behavior of system (1.5) with $\alpha_4 = 0$, and find all possible exact explicit parametric representations for the traveling wave solutions of system (1.5).

System (1.5) has the first integrals as follows: for $\beta \neq -1, -2$,

$$H(\phi, y) = y^{2}(\phi^{2} - a_{0})^{\beta} + \frac{1}{(\beta+1)(\beta+2)} [\alpha_{4}\beta(\beta+1)\phi^{4} + \beta(\beta\alpha_{2} + 2\alpha_{4}a_{0} + 2\alpha_{2})\phi^{2} + (\alpha_{0}\beta^{2} + 3\alpha_{0}\beta + a_{0}\alpha_{2}\beta + 2a_{0}\alpha_{2} + 2a_{0}^{2}\alpha_{4} + 2\alpha_{0})](\phi^{2} - a_{0})^{\beta} = h,$$
(1.6)

for $\beta = -1$,

$$H(\phi, y) = \frac{y^2}{\phi^2 - a_0} - \alpha_4 \phi^2 + \frac{a_0^2 \alpha_4 + a_0 \alpha_2 + \alpha_0}{\phi^2 - a_0} - \ln \left| \frac{\alpha_2}{\alpha_0^2 \alpha_4^2 (\phi^2 - a_0)} \right| = h, \quad (1.7)$$

for $\beta = -2$,

$$H(\phi, y) = \frac{y^2}{(\phi^2 - a_0)^2} + \frac{2\alpha_2 + 4a_0\alpha_4}{\phi^2 - a_0} + \frac{a_0^2\alpha_4 + a_0\alpha_2 + \alpha_0}{(\phi^2 - a_0)^2} - 2\alpha_4 \ln|\phi^2 - a_0| = h.$$
(1.8)

Clearly, for $a_0 > 0$, system (1.5) is a singular nonlinear traveling wave system of the first class defined in [4,5] and [7] with two singular straight lines $\phi = \pm \sqrt{a_0}$. It is very interesting that singular traveling systems have peakon, pseudo-peakon, periodic peakon and compacton solution families. Periodic peakon is classical solution with two time scales of a singular traveling system. Peakon is a limit solution of a family of periodic peakons or a limit solution of a family of pseudo-peakons under two classes of limit senses (see [8]). Compacton family is a solution family of system (1.5) for which all solutions $\phi(\xi)$ have finite support set, i.e., the defined region of every $\phi(\xi)$ with respect to ξ is finite and the value region of ϕ is bounded. Corresponding to different types of phase orbits, in [4,5] and [7], the authors derived a classification for different wave profiles of $\phi(\xi)$.

In this paper, we use the method of dynamical systems to investigate the dynamical behavior of system (1.5) with $\alpha_4 \neq 0$ and find all possible exact explicit parametric representations for all bounded solutions $\phi(\xi)$ of system (1.5). We shall see that the solutions of system (1.5) have very abounded dynamical behavior.

The main result of this paper is the following Theorem.

Theorem 1.1. Suppose that $a_0\beta\alpha_0\alpha_2\alpha_4 \neq 0$.

(1) For the five-parameter system (1.5), under 28 different parameter conditions, it has 28 different phase portraits given by Figure 1-Figure 5.

(2) Corresponding to different level curves defined by $H(\phi, y) = h$ in (1.6) with $\beta = 1, 2, -3, -4$, system (1.5) has at least 62 different exact explicit solutions $\phi(\xi)$ given by (3.3)-(4.11) and (5.3)-(6.7). These solutions give rise to 62 different exact explicit solutions with the form (1.3) for equation (1.2).

(3) System (1.5) has 18 exact explicit solitary wave solutions given by (3.5), (3.11), (3.14), (3.20), (3.34) and (4.1) for $\beta = 1$; (3.7), (3.13), (3.15), (3.21) and (3.35) for $\beta = 2$; (5.5), (5.12) and (6.2) for $\beta = -3$; (5.10), (5.13), (6.3) and (6.5) for $\beta = -4$.

(4) System (1.5) has 11 exact explicit kink and anti-kink wave solutions given by (3.3), (3.10), (4.2), (4.5) and (4.8) for $\beta = 1$; (3.9), (4.5) and (4.9) for $\beta = 2$; (5.4) for $\beta = -3$; (5.9) and (6.6) for $\beta = -4$.

(5) System (1.5) has 2 exact explicit solitary cusp wave solutions (peakons) given by (3.22) for $\beta = 1$ and (3.23) for $\beta = 2$.

(6) System (1.5) has 4 exact explicit periodic peakon solutions given by (3.24) and (4.7) for $\beta = 1$; (3.25) and (4.7) for $\beta = 2$.

(7) System (1.5) has 22 exact explicit periodic wave solutions given by (3.12), (3.16), (3.18), (3.26), (3.31), (3.32), (4.10) and (4.11) for $\beta = 1$; (3.17), (3.18), (3.27), (3.33), (4.10) and (4.11) for $\beta = 2$; (5.3), (5.6), (5.7), (5.8), (5.11), (6.1) and (6.7) for $\beta = -3$; (6.4) for $\beta = -4$.

(8) For $a_0 > 0$, corresponding to the open orbit families of system (1.5) which when $|y| \to \infty$ tend to two singular straight lines $\phi = \pm \sqrt{a_0}$, there exist a lot of compacton families of system (1.5).

The proof of this theorem is given in Section 2–6.

This paper is organized as follows. In section 2, we discuss bifurcations of phase portraits of system (1.5). In section 3-6, we consider the exact solutions of system (1.5) under different parameter conditions.

2. Bifurcations of phase portraits of system (1.5)

In this section, we consider bifurcations of phase portraits of system (1.5) depending on the parameter group $(a_0, \beta, \alpha_0, \alpha_2, \alpha_4)$. We study the associated regular system of system (1.5) as follows:

$$\frac{d\phi}{d\zeta} = y(a_0 - \phi^2), \quad \frac{dy}{d\zeta} = \beta\phi(y^2 + \alpha_0 + \alpha_2\phi^2 + \alpha_4\phi^4), \quad (2.1)$$

where $d\xi = (a_0 - \phi^2)d\zeta$, for $a_0 - \phi^2 \neq 0$. System (2.1) has the same first integrals as (1.6) - (1.8). But when $a_0 > 0$, the vector fields defined by system (2.1) and system (1.5) are different (see [6,7]).

We assume that $\alpha_0 \alpha_2 \alpha_4 \neq 0, \beta \neq -1, -2$. Clearly, system (2.1) always has the equilibrium point $E_0(0,0)$. Write that $f(p) = \alpha_0 + \alpha_2 p + \alpha_4 p^2$.

(i) When $\Delta = \alpha_2^2 - 4\alpha_0\alpha_4 > 0$, and $\alpha_0\alpha_4 > 0, \alpha_2\alpha_4 < 0$, there exist two positive equilibrium points $E_1(\phi_1, 0)$ and $E_2(\phi_2, 0)$ of system (2.1), where $\phi_1 < \phi_2$ and when $\alpha_2 < 0, \alpha_4 > 0, \phi_1^2 = \frac{1}{2\alpha_4}(-\alpha_2 - \sqrt{\Delta}), \phi_2^2 = \frac{1}{2\alpha_4}(-\alpha_2 + \sqrt{\Delta})$; when $\alpha_2 > 0, \alpha_4 < 0, \phi_2^2 = \frac{1}{2\alpha_4}(-\alpha_2 - \sqrt{\Delta}), \phi_1^2 = \frac{1}{2\alpha_4}(-\alpha_2 + \sqrt{\Delta})$. (ii) When $\Delta > 0, \alpha_0\alpha_4 < 0$, system (2.1) has only one positive equilibrium point

(ii) When $\Delta > 0$, $\alpha_0 \alpha_4 < 0$, system (2.1) has only one positive equilibrium point $E_1(\phi_e, 0)$ where $\phi_e^2 = \frac{1}{2\alpha_4}(-\alpha_2 + \sqrt{\Delta})$, if $\alpha_4 > 0$ or $\phi_e^2 = \frac{1}{2\alpha_4}(-\alpha_2 - \sqrt{\Delta})$, if $\alpha_4 < 0$. (iii) When $\Delta = 0$ and $\alpha_2 \alpha_4 < 0$, system (2.1) has a positive double equilibrium point $E_d\left(\sqrt{-\frac{\alpha_2}{2\alpha_4}}, 0\right)$.

In addition, when $a_0 > 0$, if $f(a_0) < 0$, then, there are four equilibrium points $E_{sj}(\mp \sqrt{a_0}, \mp Y_s), j = 1, 2, 3, 4$ of system (2.1) in two straight lines $\phi = \mp \sqrt{a_0}$, where $Y_s = \sqrt{-f(a_0)}$.

Let $M(\phi_j, y_j)$ be the coefficient matrix of the linearized system of system (2.1) at an equilibrium point $E_j(\phi_j, y_j)$ and $J(\phi_j, y_j) = \det M(\phi_j, y_j)$. We have

$$J(0,0) = -a_0 \alpha_0 \beta, \ J(\phi_j,0) = -\beta (a_0 - \phi_j^2) (\alpha_0 + 3\alpha_2 \phi_j^2 + 5\alpha_4 \phi_j^4), \ J(\sqrt{a_0}, Y_s) = -4\beta a_0 Y_s^2,$$
$$(\operatorname{trace} M(\sqrt{a_0}, Y_s))^2 - 4J(\sqrt{a_0}, Y_s) = 4a_0 Y_s^2 (1+\beta)^2.$$

By the theory of planar dynamical systems (see [7]), for an equilibrium point of planar integrable system, if J < 0, then the equilibrium point is a saddle point:

a planar integrable system, if J < 0, then the equilibrium point is a saddle point; If J > 0 and $(\text{Trice}M)^2 - 4J < 0(>0)$, then it is a center point (a node point); if J = 0 and the Poincaré index of the equilibrium point is 0, then this equilibrium point is a cusp.

Write that $h_0 = H(0,0)$, $h_j = H(\phi_j,0)$ and $h_s = H(\sqrt{a_0}, Y_s) = 0$, for $\beta > 0$ and $h_s = \infty$ for $\beta < 0$, where $H(\phi, y)$ is defined by (1.6).

We consider the case of system (1.5) has two positive equilibrium points. Especially, for $\beta = 1, 2$ and $\beta = -3, -4$, we can get exact explicit parametric representations for the solitary wave solutions, kink and anti-kink wave solutions and some periodic solutions.

When $\beta = 1, 2$ and $\beta = -3, -4$, the Hamiltonian (1.6) becomes respectively as follows:

$$H_1(\phi, y) = y^2(\phi^2 - a_0) + \frac{1}{6}(\phi^2 - a_0)[2\alpha_4\phi^4 + (3\alpha_2 + 2a_0\alpha_4)\phi^2 + (6\alpha_0 + 3a_0\alpha_2 + 2a_0^2\alpha_4)] \quad (2.2)$$

$$H_{2}(\phi, y) = y^{2}(\phi^{2} - a_{0})^{2} + \frac{1}{12}(\phi^{2} - a_{0})^{2}[6\alpha_{4}\phi^{4} + 4(2\alpha_{2} + a_{0}\alpha_{4})\phi^{2} + (12\alpha_{0} + 4a_{0}\alpha_{2} + 2a_{0}^{2}\alpha_{4})]$$

$$(2.3)$$

$$H_{-3}(\phi, y) = \frac{y^2}{(\phi^2 - a_0)^3} + \frac{6\alpha_4 \phi^4 + (3\alpha_2 - 6a_0\alpha_4)\phi^2 + 2a_0^2\alpha_4 - a_0\alpha_2 + 2\alpha_0}{2(\phi^2 - a_0)^3}$$
(2.4)

and

$$H_{-4}(\phi, y) = \frac{y^2}{(\phi^2 - a_0)^4} + \frac{6\alpha_4\phi^4 + 4(\alpha_2 - a_0\alpha_4)\phi^2 + (3\alpha_0 + a_0^2\alpha_4 - a_0\alpha_2)}{3(\phi^2 - a_0)^4}.$$
 (2.5)

By using the above information to do qualitative analysis, we have the following bifurcations of the phase portraits of system (1.5) shown in Figure.1-Figure.5.

1. The case $\beta = 1, 2, \Delta > 0, \alpha_0 \alpha_4 > 0, \alpha_0 > 0, \alpha_2 < 0$. In this case, the origin $E_0(0,0)$ is a saddle point.

We first assume that $\sqrt{a_0} > \phi_2$. Then, when $h_2 < h_0 < h_1$, we have phase portrait Figure.1 (a_1) . When $h_2 = h_0 < h_1$, we have phase portrait Figure.2 (a_2) . When $h_0 < h_2 < h_1$, we have phase portrait Figure.2 (a_3) . For three fixed threeparameter groups $(\alpha_0, \alpha_2, \alpha_4)$ satisfying the conditions of case 1, we decrease the parameter a_0 , then the first fixed three-parameter group gives the bifurcations of the phase portraits of system (1.5) shown in Figure.1 (a_1) -(g).



Figure.1 The bifurcations of phase portraits of system (1.5) for $\beta = 1, 2$

 $\begin{array}{l} (a_1) \ \sqrt{a_0} > \phi_2, h_2 < h_0 < h_1. \ (b_1) \ \sqrt{a_0} = \phi_2, Y_s = 0, h_2 = 0 < h_0 < h_1. \ (c) \\ \phi_1 < \sqrt{a_0} < \phi_2, h_2 < 0 < h_0 < h_1. \ (d) \ \phi_1 < \sqrt{a_0} < \phi_2, h_2 < 0 = h_0 < h_1. \ (e) \\ \phi_1 < \sqrt{a_0} < \phi_2, h_2 < h_0 < 0 < h_1. \ (f) \ \sqrt{a_0} = \phi_1, h_2 < h_0 < 0 = h_1, Y_s = 0. \ (g) \\ \sqrt{a_0} < \phi_1, h_2 < h_0 < h_1. \ The values of \ h_0, h_1, h_2 \ are obtained with \ \beta = 1. \end{array}$

The second fixed three-parameter group gives the bifurcations of the phase portraits of system (1.5) shown in Figure 2 $(a_2), (b_2)$ and Figure 1 (e)-(g). The third fixed three-parameter group gives the bifurcations of the phase portraits of system (1.5) shown in Figure 2 $(a_3), (b_3)$ and Figure 1 (e)-(g).



Figure.2 The bifurcations of phase portraits of system (1.5) for $\beta = 1, 2$

 $(a_2) \sqrt{a_0} > \phi_2, h_2 = h_0 < h_1 (b_2) \sqrt{a_0} = \phi_2, h_2 = h_0 = 0 < h_1. (a_3) \sqrt{a_0} > \phi_2, h_0 < h_2 < h_1 (b_3) \sqrt{a_0} = \phi_2, h_0 < h_2 = 0 < h_1$. The values of h_0, h_1, h_2 are obtained with $\beta = 1$.

2. The case $\beta = 1, 2, \Delta > 0, \alpha_0 \alpha_4 > 0, \alpha_0 < 0, \alpha_2 > 0$. In this case, the origin $E_0(0,0)$ is a center point.

For a fixed three-parameter group $(\alpha_0, \alpha_2, \alpha_4)$ satisfying the conditions in case 2, we vary the parameter a_0 from $\sqrt{a_0} > \phi_2$ to $\sqrt{a_0} < \phi_1$. Then, we have the bifurcations of phase portraits of system (1.5) shown in Figure.3 (a)-(f).



Figure. 3 The bifurcations of phase portraits of system (1.5) for $\beta = 1, 2$

(a) $\phi_2 < \phi_M < \sqrt{a_0}, 0 < h_1 < h_0 < h_2$. (b) $\phi_2 < \sqrt{a_0}, h_1 = 0 < h_0 < h_2$. (c) $\phi_2 < \sqrt{a_0}, h_1 < h_0 < 0 < h_2$. (d) $\sqrt{a_0} = \phi_2, h_1 < h_0 < 0 = h_2$. (e) $\phi_1 < \sqrt{a_0} < \phi_2, h_1 < h_0 < 0 < h_2$. (f) $\sqrt{a_0} = \phi_1, h_1 = 0 < h_0 < h_2$. (g) $0 < \sqrt{a_0} < \phi_1, h_1 < 0 < h_0 < h_2$. ϕ_M is the ϕ -coordinate of the right homoclinic orbit passing through the positive ϕ -axis. The values of h_0, h_1, h_2 are obtained with $\beta = 1$.

3. The case $\beta = -3, -4, \Delta > 0, \alpha_0 \alpha_4 > 0, \alpha_0 > 0, \alpha_2 < 0$. In this case, the origin $E_0(0,0)$ is a center point.

For a fixed three-parameter group $(\alpha_0, \alpha_2, \alpha_4)$ satisfying the conditions in case 3, we vary the parameter a_0 from $\sqrt{a_0} > \phi_2$ to $\sqrt{a_0} < \phi_1$. Then, we have the bifurcations of phase portraits of system (1.5) shown in Figure 4 (a)-(e).





Figure. 4 The bifurcations of phase portraits of system (1.5) for $\beta = -3, -4$

(a) $\sqrt{a_0} > \phi_2, h_1 < h_0 < h_2 < 0$. (b) $\sqrt{a_0} = \phi_2, h_1 < h_0 < 0 < h_2 = \infty$. (c) $\phi_1 < \sqrt{a_0} < \phi_2, h_1 < h_0 < 0 < h_2$. (d) $\sqrt{a_0} = \phi_1, h_0 < 0 < h_2$. (e) $0 < \sqrt{a_0} < \phi_1, h_0 < h_1 < 0 < h_2$. The values of h_0, h_1, h_2 are obtained with $\beta = -3$.

4. The case $\beta = -3, -4, \Delta > 0, \alpha_0 \alpha_4 > 0, \alpha_0 < 0, \alpha_2 > 0$. In this case, the origin $E_0(0,0)$ is a saddle point.

For a fixed three-parameter group $(\alpha_0, \alpha_2, \alpha_4)$ satisfying the conditions in case 4, we vary the parameter a_0 from $\sqrt{a_0} > \phi_2$ to $\sqrt{a_0} < \phi_1$. Then, we have the bifurcations of phase portraits of system (1.5) shown in Figure. 5 (a)-(e).



Figure. 5 The bifurcations of phase portraits of system (1.5) for $\beta = -3, -4$

(a) $\sqrt{a_0} > \phi_2, h_2 < h_0 < h_1$. (b) $\sqrt{a_0} = \phi_2, h_0 < h_1$. (c) $\phi_1 < \sqrt{a_0} < \phi_2, h_2 < 0 < h_0 < h_1$. (d) $\sqrt{a_0} = \phi_1, h_2 < 0 < h_0$. (e) $0 < \sqrt{a_0} < \phi_1, h_2 < 0 < h_1 < h_0$. The values of h_0, h_1, h_2 are obtained with $\beta = -3$.

3. Exact parametric representations of solutions of system (1.5) when $\beta = 1, 2$ in Figure.1 and Figure.2

In this section, we discuss possible parametric representations of the level curves defined by $H_j(\phi, y) = h, j = 1, 2$ in (2.2) and (2.3) for the case 1 in section 2. In this case, the parameter conditions $\Delta > 0, \alpha_0 \alpha_4 > 0, \alpha_0 > 0, \alpha_2 < 0$ hold.

Notice that for $j = 1, 2, H_j(\phi, y) = h$ can be written as

$$y^{2} = \frac{A_{0} - \alpha_{0}\phi^{2} - \frac{1}{2}\alpha_{2}\phi^{4} - \frac{1}{3}\alpha_{4}\phi^{6}}{\phi^{2} - a_{0}} \equiv \frac{F_{6}(\phi)}{\phi^{2} - a_{0}}, \quad A_{0} = h + a_{0}\alpha_{0} + \frac{1}{2}\alpha_{2}a_{0}^{2} + \frac{1}{3}\alpha_{4}a_{0}^{3}.$$
(3.1)

and

$$y^{2} = \frac{B_{0} + 2a_{0}\alpha_{0}\phi^{2} + (a_{0}\alpha_{2} - \alpha_{0})\phi^{4} + \frac{2}{3}(a_{0}\alpha_{4} - \alpha_{2})\phi^{6} - \frac{1}{2}\alpha_{4}\phi^{8}}{(\phi^{2} - a_{0})^{2}} \equiv \frac{F_{8}(\phi)}{(\phi^{2} - a_{0})^{2}},$$
(3.2)

where $B_0 = h - a_0^2 \alpha_0 - \frac{1}{3} \alpha_2 a_0^3 - \frac{1}{6} \alpha_4 a_0^4$.

By using the first equation of system (1.5), we know that $\xi = \int_{\phi_0}^{\phi} \sqrt{\frac{\phi^2 - a_0}{F_6(\phi)}} d\phi$ and $\xi = \int_{\phi_0}^{\phi} \frac{|\phi^2 - a_0| d\phi}{\sqrt{F_8(\phi)}}$. Obviously, if and only if polynomial $F_6(\phi)$ and $F_8(\phi)$ can be decomposed into a product of quadratic factors, then the two integrals can be solved.

3.1. The case of Figure 1 (a_1) .

(i) For $\beta = 1$, corresponding to the two heteroclinic orbits defined by $H_1(\phi, y) = h_2$, we have $y^2 = \frac{\alpha_4(\phi_2^2 - \phi^2)^2(\phi^2 + r_1^2)}{3(a_0 - \phi^2)}$, where $r_1^2 = \frac{\alpha_2 + 2\sqrt{\Delta}}{2\alpha_4}$. By using the first equation of system (1.5), we obtain $\sqrt{\frac{4\alpha_4}{3}} \xi = \int_0^u \frac{(a_0 - u)du}{(\phi_2^2 - u)\sqrt{(a_0 - u)u(u + r_1^2)}}$, where $u = \phi^2$. It gives rise to the following parametric representations of kink and anti-kink wave solutions of system (1.5):

$$\begin{split} \phi(\chi) &= \pm \frac{k r_1 \operatorname{sn}(\chi, k)}{\operatorname{dn}(\chi, k)}, \quad \chi \in \left(-\operatorname{sn}^{-1}\left(\frac{\phi_2}{k\sqrt{r_1^2 + \phi_2^2}}, k\right), \operatorname{sn}^{-1}\left(\frac{\phi_2}{k\sqrt{r_1^2 + \phi_2^2}}, k\right) \right), \\ \xi(\chi) &= \frac{\sqrt{\frac{3}{a_4}}}{(r_1^2 + \phi_2^2)} \left[\sqrt{a_0 + r_1^2} \chi + \frac{r_1^2(a_0 - \phi_2^2)}{\phi_2^2\sqrt{a_0 + r_1^2}} \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \tilde{\alpha}_1^2, k) \right], \end{split}$$
(3.3)

where $\tilde{\alpha}_1^2 = k^2 \left(1 + \frac{r_1^2}{\phi_2^2} \right), k^2 = \frac{a_0}{a_0 + r_1^2}.$

Corresponding to the unstable manifold of the saddle point $E_2(\phi_2, 0)$ given by $H_1(\phi, y) = h_2$ in the upper phase plane tending to the straight line $\phi = \sqrt{a_0}$ when $y \to \infty$, we have $\sqrt{\frac{4\alpha_4}{3}} \xi = \int_u^{a_0} \frac{(a_0 - u)du}{(u - \phi_2^2)\sqrt{(a_0 - u)u(u + r_1^2)}}$. Thus, we obtain the parametric representation of a bounded solution of system (1.5) as follows:

$$\phi(\chi) = \sqrt{a_0} \operatorname{cn}(\chi, k), \quad \chi \in \left(-\operatorname{cn}^{-1}\left(\frac{\phi_2}{\sqrt{a_0}}, k\right), 0\right), \\
\xi(\chi) = \sqrt{\frac{3}{\alpha_4(a_0 + r_1^2)}} (r_1^2 + \phi_2^2) \left[-\chi + \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \tilde{\alpha}_2^2, k)\right],$$
(3.4)

where $\tilde{\alpha}_2^2 = \frac{a_0}{a_0 - \phi_2^2}, k^2 = \frac{a_0}{a_0 + r_1^2}.$

The level curves defined by $H_1(\phi, y) = h_0$ are two homoclinic orbits and two open curves which passing through ϕ -axis at the points ($\pm r_1, 0$) and tending to the straight lines $\phi = \pm \sqrt{a_0}$ when $|y| \to \infty$, respectively, where $h_0 = -\frac{1}{6}a_0(6\alpha_0 + \frac{1}{6}\alpha_0)$ $3a_0\alpha_2 + 2a_0^2\alpha_4).$ Corresponding to the two homoclinic orbits, we have $y^2 = \frac{\alpha_4\phi^2(r_1^2 - \phi^2)(\phi_M^2 - \phi^2)}{3(a_0 - \phi^2)},$ where $r_1^2 = \frac{3|\alpha_2| + \sqrt{9\alpha_2^2 - 48\alpha_0\alpha_4}}{4\alpha_4}, \phi_M^2 = \frac{3|\alpha_2| - \sqrt{9\alpha_2^2 - 48\alpha_0\alpha_4}}{4\alpha_4}.$ Hence, we get $\sqrt{\frac{4\alpha_4}{3}}\xi = \int_u^{\phi_M^2} \frac{(a_0 - u)du}{u\sqrt{(a_0 - u)(r_1^2 - u)(\phi_M^2 - u)}}.$ It give rise to the following solitary wave solutions of system (1.5):

$$\begin{aligned} \phi(\chi) &= \pm \left(r_1^2 - \frac{r_1^2 - \phi_M^2}{\operatorname{cn}^2(\chi, k)} \right)^{\frac{1}{2}}, \quad \chi \in \left(-\operatorname{cn}^{-1} \left(\sqrt{1 - \frac{\phi_M^2}{r_1^2}}, k \right), \operatorname{cn}^{-1} \left(\sqrt{1 - \frac{\phi_M^2}{r_1^2}}, k \right) \right), \\ \xi(\chi) &= \frac{\sqrt{\frac{3}{\alpha_4}}}{r_1^2 \sqrt{a_0 - \phi_M^2}} \left[(a_0 - r_1^2) \chi + \frac{a_0 (r_1^2 - \phi_M^2)}{\phi_M^2} \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \tilde{\alpha}_3^2, k) \right], \end{aligned}$$

$$(3.5)$$

where $\tilde{\alpha}_3^2 = \frac{r_1^2}{\phi_M^2}, k^2 = \frac{a_0 - r_1^2}{a_0 - \phi_M^2}$. Corresponding to the open curves passing through the points $(\pm r_1, 0)$, we have $\sqrt{\frac{4\alpha_4}{3}}\xi = \int_u^{a_0} \frac{(a_0 - u)du}{u\sqrt{(a_0 - u)(u - r_1^2)(u - \phi_M^2)}}$. Therefore, we obtain the following two compacton solutions:

$$\phi(\chi) = \pm \left(a_0 - (a_0 - r_1^2) \operatorname{sn}^2(\chi, k)\right)^{\frac{1}{2}}, \quad \chi \in (-K(k), K(k)),
\xi(\chi) = \frac{\sqrt{\frac{3}{\alpha_4}}}{\sqrt{a_0 - \phi_M^2}} \left[-\chi + \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \tilde{\alpha}_4^2, k)\right],$$
(3.6)

where $k^2 = \frac{a_0 - r_1^2}{a_0 - \phi_M^2}$, $\tilde{\alpha}_4^2 = \frac{a_0 - r_1^2}{a_0}$. (ii) For $\beta = 2$, the level curves defined by $H_2(\phi, y) = h_0$ are two homoclinic orbits and four open curves which passing through ϕ -axis at the points $(\pm r_1, 0), (\pm r_2, 0), r_2 < \sqrt{a_0} < r_1$ and tending to the straight lines $\phi = \pm \sqrt{a_0}$ when $|y| \to \infty$, respectively, where $h_0 = -\frac{1}{6}a_0^2(6\alpha_0 + 2a_0\alpha_2 + a_0^2\alpha_4)$.

Corresponding to two homoclinic orbits, we have $\sqrt{2\alpha_4}\xi = \int_u^{\phi_M^2} \frac{(a_0-u)du}{u\sqrt{(r_1^2-u)(r_2^2-u)(\phi_M^2-u)}}$. We have the following solitary wave solutions of system (1.5):

$$\phi(\chi) = \pm \left(r_2^2 - \frac{r_2^2 - \phi_M^2}{\operatorname{cn}^2(\chi, k)} \right)^{\frac{1}{2}}, \quad \chi \in \left(-\operatorname{cn}^{-1} \left(\sqrt{1 - \frac{\phi_M^2}{r_2^2}}, k \right), \operatorname{cn}^{-1} \left(\sqrt{1 - \frac{\phi_M^2}{r_2^2}}, k \right) \right),$$

$$\xi(\chi) = \frac{\sqrt{2}}{r_2^2 \sqrt{\alpha_4(r_1^2 - \phi_M^2)}} \left[(a_0 - r_2^2)\chi + \frac{a_0(r_2^2 - \phi_M^2)}{\phi_M^2} \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \tilde{\alpha}_5^2, k) \right],$$
(3.7)

where $\tilde{\alpha}_5^2 = \frac{r_2^2}{\phi_M^2}, k^2 = \frac{r_1^2 - r_2^2}{r_1^2 - \phi_M^2}$. Corresponding to the open curves passing through the points $(\pm r_1, 0)$, we have $\sqrt{2\alpha_4}\xi = \int_u^{r_1^2} \frac{(u-a_0)du}{u\sqrt{(r_1^2 - u)(u - r_2^2)(u - \phi_M^2)}}$. Therefore, we obtain the following two compacton solutions:

$$\begin{aligned} \phi(\chi) &= \pm \left(r_1^2 - (r_1^2 - r_2^2) \operatorname{sn}^2(\chi, k) \right)^{\frac{1}{2}}, \chi \in \left(-\operatorname{sn}^{-1} \left(\sqrt{\frac{r_1^2 - a_0}{r_1^2 - r_2^2}}, k \right), \operatorname{sn}^{-1} \left(\sqrt{\frac{r_1^2 - a_0}{r_1^2 - r_2^2}}, k \right) \right), \\ \xi(\chi) &= \frac{\sqrt{2}}{r_1^2 \sqrt{\alpha_4(r_1^2 - \phi_M^2)}} \left[(r_1^2 \chi - a_0 \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \tilde{\alpha}_6^2, k) \right], \end{aligned}$$

$$(3.8)$$

where $k^2 = \frac{r_1^2 - r_2^2}{r_1^2 - \phi_M^2}$, $\tilde{\alpha}_6^2 = \frac{r_1^2 - r_2^2}{r_1^2}$. Similarly, corresponding to the open curves passing through the points $(\pm r_2, 0)$, we can obtain two compacton solutions.

The level curves defined by $H_2(\phi, y) = h_2$ are two heteroclinic orbits connecting the equilibrium points $(\pm \phi_2, 0)$ and two open curves which passing through ϕ -axis at the points $(\pm r_1, 0), a_0 < r_1^2$ and tending to the straight lines $\phi = \pm \sqrt{a_0}$ when $|y| \to \infty$, respectively. For two heteroclinic orbits, we have $\sqrt{2\alpha_4}\xi = \int_0^u \frac{(a_0-u)du}{(\phi_2^2-u)\sqrt{(r_1^2-u)u(u+r_2^2)}}$. It gives rise to the following parametric representations of kink and anti-kink wave solutions of system (1.5):

$$\phi(\chi) = \pm \frac{kr_2 \operatorname{sn}(\chi,k)}{\operatorname{dn}(\chi,k)}, \quad \chi \in \left(-\operatorname{sn}^{-1}\left(\frac{\phi_2}{k\sqrt{r_2^2 + \phi_2^2}}, k\right), \operatorname{sn}^{-1}\left(\frac{\phi_2}{k\sqrt{r_1^2 + \phi_2^2}}, k\right)\right), \\
\xi(\chi) = \frac{\sqrt{2}}{(r_2^2 + \phi_2^2)\sqrt{\alpha_4}(r_1^2 + r_2^2)} \left[\sqrt{a_0 + r_2^2}\chi + \frac{r_2^2(a_0 - \phi_2^2)}{\phi_2^2}\Pi(\operatorname{arcsin}(\operatorname{sn}(\chi,k)), \tilde{\alpha}_7^2, k)\right], \\
(3.9)$$

where $\tilde{\alpha}_7^2 = k^2 \left(1 + \frac{r_2^2}{\phi_2^2} \right), k^2 = \frac{r_1^2}{r_1^2 + r_2^2}.$

3.2. The case of Figure 2 (a_2) .

(i) For $\beta = 1$, corresponding to the four heteroclinic orbits defined by $H_1(\phi, y) = h_2 = h_0$ connecting the equilibrium points $(\pm \phi_2, 0)$ and (0, 0), we have $\sqrt{\frac{\alpha_4}{3}}\xi = \int_{\phi_1}^{\phi} \frac{\sqrt{a_0 - \phi^2} d\phi}{\phi(\phi_2^2 - \phi^2)} = \int_{\phi_1}^{\phi} \frac{d\phi}{\phi\sqrt{a_0 - \phi^2}} + (a_0 - \phi_2^2) \int_{\phi_1}^{\phi} \frac{d\phi}{\phi(\phi_2^2 - \phi^2)\sqrt{a_0 - \phi^2}}$. thus, we obtain the following parametric representation of the right heteoclinic orbit of system (1.5):

$$\begin{aligned} \phi(\chi) &= \frac{4a_0 P_1}{P_1^2 e^{\sqrt{a_0 \chi}} + 4a_0 e^{-\sqrt{a_0 \chi}}}, \\ \xi(\chi) &= \sqrt{\frac{3}{\alpha_4}} \left[\chi + \frac{\sqrt{a_0 - \phi_2^2}}{2\phi_2^2} \left(\ln \frac{\left(\sqrt{a_0 - \phi_2^2} \sqrt{a_0 - \phi^2(\chi)} + a_0\right)^2 - \phi_2^2 \phi^2(\chi)}{\phi_2^2 - \phi^2(\chi)} \right. \\ \left. - 2\sqrt{1 - \frac{\phi_2^2}{a_0}} \ln \frac{\sqrt{a_0(a_0 - \phi^2(\chi))} + a_0}{\phi(\chi)} - B_0 \right) \right], \end{aligned}$$
(3.10)

where

$$\begin{split} P_1 &= \frac{2\sqrt{a_0(a_0-\phi_1^2+2a_0}}{\phi_1}, \\ B_0 &= \ln \frac{\left(\sqrt{a_0-\phi_2^2}\sqrt{a_0-\phi_1^2}+a_0\right)^2 - \phi_2^2\phi_1^2}{\phi_2^2 - \phi_1^2} - 2\sqrt{1-\frac{\phi_2^2}{a_0}}\ln \frac{\sqrt{a_0(a_0-\phi_1^2)}+a_0}{\phi_1}. \end{split}$$

(ii) For $\beta = 2$, corresponding to the four heteroclinic orbits defined by $H_2(\phi, y) = h_2 = h_0$ connecting the equilibrium points $(\pm \phi_2, 0)$ and (0, 0), we have $y^2 = \frac{\alpha_4(r_1^2 - \phi^2)(\phi_2^2 - \phi^2)^2 \phi^2}{2(a_0 - \phi^2)^2}$. Thus, from $\sqrt{\frac{\alpha_4}{2}} \xi = \int_{\phi_1}^{\phi} \frac{(a_0 - \phi^2)d\phi}{\phi(\phi_2^2 - \phi^2)\sqrt{r_1^2 - \phi^2}} = \int_{\phi_1}^{\phi} \frac{d\phi}{\phi\sqrt{r_1^2 - \phi^2}} + (a_0 - \phi_2^2) \int_{\phi_1}^{\phi} \frac{d\phi}{\phi(\phi_2^2 - \phi)\sqrt{r_1^2 - \phi^2}}$, we obtain the similar parametric representation of the right heteoclinic orbit of system (1.5) as (3.10), where a_0 be changed to $r_1^2, \sqrt{\frac{3}{\alpha_4}}$ be changed to $\sqrt{\frac{2}{\alpha_4}}$.

3.3. The case of Figure 2 (a_3) .

(i) For $\beta = 1$, corresponding to the two homoclinic orbits defined by $H_1(\phi, y) = h_2$ to the equilibrium points $(\pm \phi_2, 0)$, we have $y^2 = \frac{\alpha_4(\phi_2^2 - \phi^2)^2(\phi^2 - \phi_m^2)}{3(a_0 - \phi^2)}$. Hence, from

 $\sqrt{\frac{\alpha_4}{3}}\xi = \int_{\phi_m}^{\phi} \frac{(a_0 - \phi^2)d\phi}{(\phi_2^2 - \phi^2)\sqrt{(a_0 - \phi^2)(\phi^2 - \phi_m^2)}}$, we obtain the following parametric representation tations of the two homoclinic orbits of system (1.5):

$$\begin{split} \phi(\chi) &= \pm \frac{\phi_m}{\mathrm{dn}(\chi,k)}, \quad \chi \in \left(-\mathrm{dn}^{-1}\left(\frac{\phi_m}{\phi_2},k\right), \mathrm{dn}^{-1}\left(\frac{\phi_m}{\phi_2},k\right)\right), \\ \xi(\chi) &= \sqrt{\frac{3}{a_0\alpha_4}} \left[\frac{a_0}{\phi_2^2}\chi + \frac{\phi_m^2(a_0-\phi_2^2)}{\phi_2^2(\phi_2^2-\phi_m^2)}\Pi(\mathrm{arcsin}(\mathrm{sn}(\chi,k)),\tilde{\alpha}_8^2,k)\right], \end{split}$$
(3.11)

where $k^2 = \frac{a_0 - \phi_m^2}{a_0}$, $\tilde{\alpha}_8^2 = \frac{k^2 \phi_2^2}{\phi_2^2 - \phi_m^2}$. (ii) For $\beta = 2$, corresponding to the two homoclinic orbits defined by $H_2(\phi, y) = h_2$ to the equilibrium points $(\pm \phi_2, 0)$, we have $y^2 = \frac{\alpha_4 (r_1^2 - \phi^2) (\phi_2^2 - \phi^2)^2 (\phi^2 - \phi_m^2)}{2(a_0 - \phi^2)^2}$, $\phi_m < \phi_2 < \sqrt{a_0} < r_1$. Hence, from $\sqrt{\frac{\alpha_4}{2}} \xi = \int_{\phi_m}^{\phi} \frac{(a_0 - \phi^2) d\phi}{(\phi_2^2 - \phi^2) \sqrt{(r_1^2 - \phi^2) (\phi^2 - \phi_m^2)}}$, we obtain the similar parametric representations of the two homoclinic orbits of system (1.5)as (3.11).

3.4. The case of Figure 1 (b_1) .

(i) For $\beta = 1$, the level curve defined by $H_1(\phi, y) = h_2 = 0$ is a closed orbit which contacts to two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \sqrt{a_0}, 0)$. We have $y^2 = \frac{1}{3}\alpha_4(a_0 - \phi^2)(\phi^2 + r_1^2), \sqrt{\frac{\alpha_4}{3}}\xi = \int_0^{\phi} \frac{d\phi}{\sqrt{(a_0 - \phi^2)(\phi^2 + r_1^2)}}$. Thus, we obtain the following parametric representation of periodic solution of system (1.5):

$$\phi(\xi) = \frac{kr_1 \operatorname{sn}(\omega_1 \xi, k)}{\operatorname{dn}(\omega_1 \xi, k)},\tag{3.12}$$

where $k^2 = \frac{a_0}{a_0 + r_1^2}, \omega_1 = \sqrt{\frac{\alpha_4(a_0 + r_1^2)}{3}}.$

Corresponding to the two homoclinic orbits defined by $H_1(\phi, y) = h_0$ we have the similar parametric representations of solitary wave solutions of system (1.5) as (3.5).

(ii) For $\beta = 2$, the level curve defined by $H_2(\phi, y) = h_2 = 0$ is a closed orbit which contacts to two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \sqrt{a_0}, 0)$. We have $y^2 = \frac{1}{2}\alpha_4(a_0 - \phi^2)(\phi^2 + r_1^2), \sqrt{\frac{\alpha_4}{2}}\xi = \int_0^\phi \frac{d\phi}{\sqrt{(a_0 - \phi^2)(\phi^2 + r_1^2)}}$. Thus, we obtain the similar parametric representation of periodic solution of system (1.5) as (3.12).

Corresponding to the two homoclinic orbits defined by $H_2(\phi, y) = h_0$, we have $y^2 = \frac{\alpha_4 \phi^2 (\phi_M^2 - \phi^2) [(\phi^2 - b_1)^2 + a_1^2]}{2(a_0 - \phi^2)^2}$. Hence, we obtain the following parametric representation of the two homoclinic orbits defined by $H_2(\phi, y) = h_0$, we have tations of two solitary wave solutions of system (1.5):

$$\begin{split} \phi(\chi) &= \pm \left(\frac{(\phi_M^2 - A_1) + (\phi_M^2 + A_1) \operatorname{cn}(\chi, k)}{1 + \operatorname{cn}(\chi, k)} \right)^{\frac{1}{2}}, \chi \in \left(-\operatorname{cn}^{-1} \left(\frac{A_1 - \phi_M^2}{A_1 + \phi_M^2}, k \right), \operatorname{cn}^{-1} \left(\frac{A_1 - \phi_M^2}{A_1 + \phi_M^2}, k \right) \right) \\ \xi(\chi) &= \frac{1}{\sqrt{2\alpha_4 A_1}} \left[\frac{1 - \phi_M^2 - A_1}{\phi_M^2 + A_1} \chi - \frac{\phi_M^2 - A_1}{2\phi_M^2 (\phi_M^2 + A_1)} \Pi \left(\operatorname{arccos}(\operatorname{cn}(\chi, k)), \frac{\tilde{\alpha}_9^2}{\tilde{\alpha}_9^2 - 1}, k \right) \right) + \frac{1}{2\phi_M^2} f_1 \right] \end{split}$$
(3.13)

where $A_1 = (b_1 - \phi_M^2)^2 + a_1, k^2 = \frac{A_1 - b_1 + \phi_M^2}{2A_1}, \tilde{\alpha}_9 = \frac{\phi_M^2 + A_1}{\phi_M^2 - A_1}$, the function f_1 can be seen in [3](361.54).

3.5. The case of Figure 2 (b_2) .

For $\beta = 1, 2$, the level curves defined by $H_1(\phi, y) = h_2 = 0$ and $H_2(\phi, y) = h_2 = 0$ are two homoclinic orbits contacting two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \sqrt{a_0}, 0), \phi_2 = \sqrt{a_0}$. In these cases, we have $y^2 = \frac{\alpha_4}{3}\phi^2(a_0 - \phi^2)$ and $y^2 = \frac{\alpha_4}{2}\phi^2(a_0 - \phi^2)$. Thus, we have the following parametric representations of two solitary wave solutions of system (1.5):

$$\phi(\xi) = \pm \sqrt{a_0} \operatorname{sech}\left(\sqrt{\frac{a_0 \alpha_4}{3}} \xi\right), \qquad (3.14)$$

and

$$\phi(\xi) = \pm \sqrt{a_0} \operatorname{sech}\left(\sqrt{\frac{a_0 \alpha_4}{2}} \xi\right).$$
(3.15)

3.6. The case of Figure 2 (b_3) .

For $\beta = 1, 2$, the level curves defined by $H_1(\phi, y) = h_2 = 0$ and $H_2(\phi, y) = h_2 = 0$ are two periodic orbits contacting two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \sqrt{a_0}, 0), \phi_2 = \sqrt{a_0}$. In these cases, we have $y^2 = \frac{\alpha_4}{3}(a_0 - \phi^2)(\phi^2 - r_m^2)$ and $y^2 = \frac{\alpha_4}{2}(a_0 - \phi^2)(\phi^2 - r_m^2)$. Thus, we have the following parametric representations of two periodic wave solutions of system (1.5):

$$\phi(\xi) = \pm \frac{r_m}{\ln\left(\sqrt{\frac{a_0\alpha_4}{3}}\xi, k\right)} \tag{3.16}$$

and

$$\phi(\xi) = \pm \frac{r_m}{\ln\left(\sqrt{\frac{a_0\alpha_4}{2}}\xi, k\right)},\tag{3.17}$$

where $k^2 = 1 - \frac{r_m^2}{a_0}$.

3.7. The case of Figure 1 (c).

(i) For $\beta = 1, 2$, the level curves defined by $H_1(\phi, y) = 0$ and $H_2(\phi, y) = 0$ are two global closed orbits enclosing five equilibrium points $(\pm \phi_1, 0), (\pm \phi_2, 0)$ and (0, 0), which pass through two straight lines $\phi = \pm \sqrt{a_0}$ and intersects the ϕ -axis at two points $(\pm r_M, 0)$, respectively. We have $y^2 = \frac{1}{3}\alpha_4(r_M^2 - \phi^2)(\phi^2 + r_1^2)$ and $y^2 = \frac{1}{2}\alpha_4(r_M^2 - \phi^2)(\phi^2 + r_1^2)$. Thus, we obtain the following parametric representations of two periodic solutions of system (1.5):

$$\phi(\xi) = r_M \operatorname{cn}(\Omega_j \xi, k), \ j = 1, 2, \ \xi \in (-\infty, \infty),$$
(3.18)

where $k^2 = \frac{r_M^2}{r_M^2 + r_1^2}, \Omega_1 = \sqrt{\frac{\alpha_4(r_M^2 + r_1^2)}{3}}, \Omega_2 = \sqrt{\frac{\alpha_4(r_M^2 + r_1^2)}{2}}.$

Notice that the two straight lines $\phi = \pm \sqrt{a_0}$ separate the inner region of the above closed curve to three areas for which there exist three period annuluses surrounding the centers $(\pm \phi_2, 0)$ and other three equilibrium points $(\pm \phi_1, 0)$ and the origin (0, 0), defined by $H_j(\phi, y) = h, j = 1, 2, h \in (h_2, 0)$ and $h \in (0, h_0)$, respectively. When $h \to 0$, these three families of periodic orbits give rise to three families of periodic peakon solutions of system (1.5). As their limit orbits, two arches define two periodic peakons with the parametric representations (see Figure.6 (b), (c)):

$$\phi(\xi) = \pm r_M \operatorname{cn}(\Omega_j \xi, k), \quad \xi \in \left(-\frac{1}{\Omega_j} \operatorname{cn}^{-1}\left(\frac{\sqrt{a_0}}{r_M}, k\right), \frac{1}{\Omega_j} \operatorname{cn}^{-1}\left(\frac{\sqrt{a_0}}{r_M}, k\right)\right). \quad (3.19_a)$$

The curve quadrangle enclosing three equilibrium points also gives rise to a periodic peakon with the parametric representations (see Figure.6 (a)):



Figure. 6 Periodic peakon and peakon solutions of system (1.5) for $\beta = 1, 2$

Corresponding to the two homoclinic orbits defined by $H_1(\phi, y) = h_0$ we have the same parametric representations of solitary wave solutions of system (1.5) as (3.5).

The level curves defined by $H_2(\phi, y) = h_0$ contain two homoclinic orbits to the origin and two periodic solutions enclosing the equilibrium points $(\pm \phi_2, 0)$. For the two homoclinic orbits, we have the same parametric representations of solitary wave solutions of system (1.5) as (3.7). For the two periodic orbits, we have the same parametric representations of periodic wave solutions of system (1.5) as (3.8) with $\chi \in (-\infty, \infty)$.

3.8. The case of Figure 1 (d).

For $\beta = 1, 2$, the level curves defined by $H_1(\phi, y) = h_0 = 0$ and $H_2(\phi, y) = h_0 = 0$ are two homoclinic orbits passing through two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \sqrt{a_0}, \pm Y_s)$. In these cases, we have $y^2 = \frac{\alpha_4}{3}\phi^2(\phi_M^2 - \phi^2)$ and $y^2 = \frac{\alpha_4}{2}\phi^2(\phi_M^2 - \phi^2)$. Thus, we have the following parametric representations of two solitary wave solutions of system (1.5):

$$\phi(\xi) = \pm \phi_M \operatorname{sech}\left(\phi_M \sqrt{\frac{\alpha_4}{3}} \xi\right), \qquad (3.20)$$

and

$$\phi(\xi) = \pm \phi_M \operatorname{sech}\left(\phi_M \sqrt{\frac{\alpha_4}{2}} \xi\right).$$
(3.21)

Notice that as a limit solution of a family of periodic orbits defined by $H_1(\phi, y) = h, h \in (0, h_1)$ and $H_2(\phi, y) = h, h \in (h_1, 0)$, enclosing the equilibrium point $(\phi_1, 0)$, there exists a curve triangle which gives rise to a peakon solution and an anti-peakon solution of system (1.5) with the parametric representations (see Figure 6 (d)):

$$\phi(\xi) = \pm \phi_M \operatorname{sech}\left(\phi_M \sqrt{\frac{\alpha_4}{3}} \xi\right), \quad \xi \in (-\infty, -\xi_{p1}), (\xi_{p1}, \infty), \tag{3.22}$$

and

$$\phi(\xi) = \pm \phi_M \operatorname{sech}\left(\phi_M \sqrt{\frac{\alpha_4}{2}} \xi\right), \quad \xi \in (-\infty, -\xi_{p2}), (\xi_{p2}, \infty), \tag{3.23}$$

where $\xi_{p1} = \frac{1}{\phi_M} \sqrt{\frac{3}{\alpha_4}} \operatorname{sech}^{-1} \left(\frac{\sqrt{a_0}}{\phi_M} \right), \xi_{p2} = \frac{1}{\phi_M} \sqrt{\frac{2}{\alpha_4}} \operatorname{sech}^{-1} \left(\frac{\sqrt{a_0}}{\phi_M} \right).$ As a limit solution of a family of periodic orbits defined by $H_1(\phi, y) = h$ and

As a limit solution of a family of periodic orbits defined by $H_1(\phi, y) = h$ and $H_2(\phi, y) = h, h \in (h_2, 0)$, enclosing the equilibrium point $(\phi_2, 0)$, there exists a curve arch which gives rise to a periodic peakon solution of system (1.5) with the parametric representations:

$$\phi(\xi) = \pm \phi_M \operatorname{sech}\left(\phi_M \sqrt{\frac{\alpha_4}{3}} \xi\right), \quad \xi \in (-\xi_{p1}, \xi_{p1}), \quad (3.24)$$

and

$$\phi(\xi) = \pm \phi_M \operatorname{sech}\left(\phi_M \sqrt{\frac{\alpha_4}{2}} \xi\right), \quad \xi \in (-\xi_{p2}, \xi_{p2}).$$
(3.25)

3.9. The case of Figure 1 (e).

(i) For $\beta = 1, 2$, the level curves defined by $H_1(\phi, y) = 0$ and $H_2(\phi, y) = 0$ are two closed orbits passing through two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \sqrt{a_0}, \pm Y_s)$. In these cases, we have $y^2 = \frac{\alpha_4}{3}(r_1^2 - \phi^2)(\phi^2 - r_2^2)$ and $y^2 = \frac{\alpha_4}{2}(r_1^2 - \phi^2)(\phi^2 - r_2^2)$. Thus, we have the following parametric representations of two periodic wave solutions of system (1.5):

$$\phi(\xi) = \pm \frac{r_2}{\ln\left(r_1 \sqrt{\frac{\alpha_4}{3}}\xi, k\right)},$$
(3.26)

and

$$\phi(\xi) = \pm \frac{r_2}{\ln\left(r_1 \sqrt{\frac{\alpha_4}{2}}\xi, k\right)},$$
(3.27)

where $k^2 = 1 - \frac{r_2^2}{r_1^2}$.

Notice that straight line $\phi = \sqrt{a_0}$ separates the above right closed orbit as two arches which are two limit solutions of two families of periodic orbits enclosing the equilibrium points $(\phi_1, 0)$ and $(\phi_2, 0)$, respectively. The two arches give rise to two periodic peakon solutions of system (1.5) with the parametric representations:

$$\phi(\xi) = \frac{r_2}{\operatorname{dn}\left(r_1\sqrt{\frac{\alpha_4}{3}}\xi, k\right)}, \xi \in \left(-\frac{K(k)}{r_1^2\sqrt{\frac{3}{\alpha_4}}}, \frac{K(k)}{r_1^2\sqrt{\frac{3}{\alpha_4}}}\right), \left(\frac{K(k)}{r_1^2\sqrt{\frac{3}{\alpha_4}}}, \frac{2K(k)}{r_1^2\sqrt{\frac{3}{\alpha_4}}}\right), \text{respectively},$$
(3.28)

and

$$\phi(\xi) = \frac{r_2}{\operatorname{dn}\left(r_1\sqrt{\frac{\alpha_4}{2}}\xi, k\right)}, \xi \in \left(-\frac{K(k)}{r_1^2\sqrt{\frac{2}{\alpha_4}}}, \frac{K(k)}{r_1^2\sqrt{\frac{2}{\alpha_4}}}\right), \left(\frac{K(k)}{r_1^2\sqrt{\frac{2}{\alpha_4}}}, \frac{2K(k)}{r_1^2\sqrt{\frac{3}{\alpha_4}}}\right), \text{respectively.}$$
(3.29)

(ii) For $\beta = 1$, the level curves defined by $H_1(\phi, y) = h_0$ are two closed orbits enclosing the equilibrium points $(\pm \phi_2, 0)$, respectively, and the stable and unstable manifolds of the saddle point (0, 0). For the unstable manifold of the origin in the upper phase plane, we have $2\sqrt{\frac{\alpha_4}{3}}\xi = \int_u^{a_0} \frac{(a_0-u)du}{u\sqrt{(r_1^2-u)(r_2^2-u)(a_0-u)}}$. It gives rise to the following bounded wave solutions of system (1.5):

$$\phi(\chi) = \left(\frac{a_0 - r_2^2 \operatorname{sn}^2(\chi, k)}{\operatorname{cn}^2(\chi, k)}\right)^{\frac{1}{2}}, \quad \chi \in \left(-\operatorname{sn}^{-1}\left(\sqrt{\frac{a_0}{r_2^2}}, k\right), 0\right),$$

$$\xi(\chi) = \frac{\sqrt{\frac{3}{a_4}}}{r_2^2 \sqrt{r_1^2 - a_0}} \left[(a_0 - r_2^2)\chi + (r_2^2 - a_0)\Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \tilde{\alpha}_{10}^2, k)\right],$$
(3.30)

where $\tilde{\alpha}_{10}^2 = \frac{r_2^2}{a_0}, k^2 = \frac{r_1^2 - r_2^2}{r_1^2 - a_0}.$

For the two periodic orbits, we have $2\sqrt{\frac{\alpha_4}{3}}\xi = \int_{r_2^2}^u \frac{(u-a_0)du}{u\sqrt{(r_1^2-u)(u-r_2^2)(u-a_0)}}$. Thus, we obtain the following parametric representations of two periodic wave solutions of system (1.5):

$$\phi(\chi) = \pm \left(a_0 + \frac{r_2^2 - a_0}{\mathrm{dn}^2(\chi, k))}\right)^{\frac{1}{2}},$$

$$\xi(\chi) = \sqrt{\frac{3}{\alpha_4(r_1^2 - a_0)}} \left(1 - \frac{a_0}{r_2^2}\right) \Pi(\arcsin(\mathrm{sn}(\chi, k)), \tilde{\alpha}_{11}^2, k),$$
(3.31)

where $k^2 = \frac{r_1^2 - r_2^2}{r_1^2 - a_0}, \tilde{\alpha}_{11}^2 = \frac{k^2 a_0}{r_2^2}.$

3.10. The case of Figure 1 (f).

(i) For $\beta = 1, 2$, the level curves defined by $H_1(\phi, y) = h_1 = 0$ and $H_2(\phi, y) = h_1 = 0, \sqrt{a_0} = \phi_1$ are two closed orbits contacting to two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \sqrt{a_0}, 0)$. In these cases, we have $y^2 = \frac{\alpha_4}{3}(r_1^2 - \phi^2)(\phi^2 - a_0)$ and $y^2 = \frac{\alpha_4}{2}(r_1^2 - \phi^2)(\phi^2 - a_0)$. Thus, we have the following parametric representations of two periodic wave solutions of system (1.5):

$$\phi(\xi) = \pm r_1 \mathrm{dn}\left(r_1 \sqrt{\frac{\alpha_4}{3}} \xi, k\right), \qquad (3.32)$$

and

$$\phi(\xi) = \pm r_1 \mathrm{dn}\left(r_1 \sqrt{\frac{\alpha_4}{2}} \xi, k\right), \qquad (3.33)$$

where $k^2 = \frac{r_1^2 - a_0}{r_1^2}$.

For $\beta = 1$, the level curves defined by $H_1(\phi, y) = h_0$ are two closed orbits enclosing the equilibrium points $(\pm \phi_2, 0)$, respectively, and the stable and unstable manifolds of the saddle point (0, 0). We have the same parametric representations as (3.30) and (3.31).

3.11. The case of Figure 1 (g).

(i) For $\beta = 1$, the level curves defined by $H_1(\phi, y) = h_1$ contain two homoclinic orbits to the equilibrium points $(\pm \phi_1, 0)$, enclosing the equilibrium points $(\pm \phi_2, 0)$, respectively. We have $2\sqrt{\frac{\alpha_4}{3}}\xi = \int_u^{\phi_M^2} \frac{(u-a_0)du}{(u-\phi_1^2)\sqrt{(\phi_M^2-u)(u-a_0)u}}$. Hence, we have the following parametric representations of two solitary wave solutions of system (1.5):

$$\phi(\chi) = \pm \phi_M dn(\chi, k), \quad \chi \in \left(-\operatorname{sn}^{-1}\left(\sqrt{\frac{\phi_M^2 - \phi_1^2}{\phi_M^2 - a_0}}, k\right), \operatorname{sn}^{-1}\left(\frac{\phi_M^2 - \phi_1^2}{\phi_M^2 - a_0}, k\right)\right), \quad (3.34)$$

$$\xi(\chi) = \frac{1}{\phi_M} \sqrt{\frac{3}{\alpha_4}} \left[\chi + \frac{\phi_1^2 - a_0}{\phi_M^2 - \phi_1^2} \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \tilde{\alpha}_{12}^2, k)\right],$$

where $k^2 = 1 - \frac{a_0}{\phi_M^2}$, $\tilde{\alpha}_{12}^2 = \frac{\phi_M^2 - a_0}{\phi_M^2 - \phi_1^2}$. (ii) For $\beta = 2$, the level curves defined by $H_2(\phi, y) = h_1$ contain two homoclinic orbits to the equilibrium points $(\pm \phi_1, 0)$, enclosing the equilibrium points $(\pm \phi_2, 0)$, respectively, and two open curves passing through the ϕ -axis at the points $(\pm r_2, 0)$ with $0 < r_2 < \sqrt{a_0} < \phi_1 < \phi_2 < \phi_M$. For two homoclinic orbits, we have $\sqrt{2\alpha_4}\xi = \int_u^{\phi_M} \frac{(u-a_0)du}{(u-\phi_1^2)\sqrt{(\phi_M^2-u)(u-r_2^2)u}}$. Therefore, we have the following parametric representations of two solitary wave solutions of system (1.5):

$$\phi(\chi) = \pm \phi_M \mathrm{dn}(\chi, k), \quad \chi \in \left(-\mathrm{sn}^{-1}\left(\sqrt{\frac{\phi_M^2 - \phi_1^2}{\phi_M^2 - r_2^2}}, k\right), \mathrm{sn}^{-1}\left(\frac{\phi_M^2 - \phi_1^2}{\phi_M^2 - r_2^2}, k\right)\right), \quad (3.35)$$

$$\xi(\chi) = \frac{1}{\phi_M} \sqrt{\frac{2}{\alpha_4}} \left[\chi + \frac{\phi_1^2 - a_0}{\phi_M^2 - \phi_1^2} \Pi(\arcsin(\mathrm{sn}(\chi, k)), \tilde{\alpha}_{13}^2, k)\right],$$

where $k^2 = 1 - \frac{r_2^2}{\phi_M^2}$, $\tilde{\alpha}_{13}^2 = \frac{\phi_M^2 - r_2^2}{\phi_M^2 - \phi_1^2}$.

4. Exact parametric representations of solutions of system (1.5) when $\beta = 1, 2$ in Figure.3

In this section, we study possible parametric representations of the level curves defined by $H_j(\phi, y) = h, j = 1, 2$, in (2.2) and (2.3) for the case 2 in section 2. In this case, the parameter conditions $\Delta > 0$, $\alpha_0 \alpha_4 > 0$, $\alpha_0 < 0$, $\alpha_2 > 0$ hold.

4.1. The case of Figure.3 (a).

(i) For $\beta = 1$, the level curves defined by $H_1(\phi, y) = h_1$ contain two homiclinic orbits to the points $(\pm \phi_1, 0)$, enclosing the centers $(\pm \phi_2, 0)$, and two heteroclinic orbits connecting two equilibrium points $(\pm \phi_1, 0)$. Corresponding to two homoclinic orbits, we have $2\sqrt{\frac{|\alpha_4|}{3}}\xi = \int_u^{\phi_M^2} \frac{(a_0-u)du}{(u-\phi_1^2)\sqrt{(a_0-u)(\phi_M^2-u)u}}$. Thus, we have the following parametric representations of two solitary wave solutions of system (1.5):

$$\begin{split} \phi(\chi) &= \pm \frac{\phi_M \operatorname{cn}(\chi, k)}{\operatorname{dn}(\chi, k)}, \quad \chi \in \left(-\operatorname{cd}^{-1}\left(\frac{\phi_1}{\phi_M}, k\right), \operatorname{cd}^{-1}\left(\frac{\phi_1}{\phi_M}, k\right) \right), \\ \xi(\chi) &= \sqrt{\frac{3}{a_0 |\alpha_4|}} \left(\frac{a_0 - \phi_M^2}{\phi_M^2 - \phi_1^2} \right) \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \hat{\alpha}_1^2, k), \end{split}$$
(4.1)

where $k^2 = \frac{\phi_M^2}{a_0}, \hat{\alpha}_1^2 = \frac{k^2(a_0 - \phi_1^2)}{\phi_M^2 - \phi_1^2}.$

Corresponding to two heteroclinic orbits, we have $2\sqrt{\frac{|\alpha_4|}{3}}\xi = \int_0^u \frac{(a_0-u)du}{(\phi_1^2-u)\sqrt{(a_0-u)(\phi_M^2-u)u}}$. Thus, we the following parametric representations of kink and anti-kink wave solutions of system (1.5):

$$\phi(\chi) = \pm \phi_M \operatorname{sn}(\chi, k), \quad \chi \in \left(-\operatorname{sn}^{-1}\left(\frac{\phi_1}{\phi_M}, k\right), \operatorname{sn}^{-1}\left(\frac{\phi_1}{\phi_M}, k\right)\right),$$

$$\xi(\chi) = \sqrt{\frac{3}{a_0|\alpha_4|}} \left[\chi + \left(1 - \frac{a_0}{\phi_1^2}\right) \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \hat{\alpha}_2^2, k)\right],$$
(4.2)

where $k^2 = \frac{\phi_M^2}{a_0}, \hat{\alpha}_2^2 = \frac{\phi_M^2}{\phi_1^2}$. For $\beta = 2$, corresponding to the level curves defined by $H_2(\phi, y) = h_1$, the above two integrals becomes $2\sqrt{\frac{|\alpha_4|}{2}}\xi = \int_u^{\phi_M^2} \frac{(a_0-u)du}{(u-\phi_1^2)\sqrt{(r_1^2-u)(\phi_M^2-u)u}}$ and $2\sqrt{\frac{|\alpha_4|}{2}}\xi =$

 $\int_0^u \frac{(a_0-u)du}{(\phi_1^2-u)\sqrt{(r_1^2-u)(\phi_M^2-u)u}}.$ Therefore, we have similar parametric representations as (4.1) and (4.2).

(ii) For $\beta = 1, 2$, the level curves defined by $H_1(\phi, y) = 0$ and $H_2(\phi, y) = 0$ contain two straight lines $\phi = \pm \sqrt{a_0}$ and two open curves passing through the points $(\pm \sqrt{a_0}, \pm Y_s)$. Now, we have $y^2 = \frac{|\alpha_4|}{3}(\phi^2 - \rho^2)(\phi^2 - \bar{\rho}^2)$ and $y^2 = \frac{|\alpha_4|}{2}(\phi^2 - \rho^2)(\phi^2 - \bar{\rho}^2)$, respectively. For the segments of the above open curve between the points $(-\sqrt{a_0}, Y_s)$ and $(\sqrt{a_0}, Y_s)$, we have the parametric representations:

$$\phi(\xi) = \pm \left(\frac{\rho \bar{\rho} (1 - \operatorname{cn}(\omega_1 \xi, k))}{(1 + \operatorname{cn}(\omega_1 \xi, k))} \right)^{\frac{1}{2}}, \xi \in \left(-\frac{1}{\omega_1} \operatorname{cn}^{-1} \left(\frac{\rho \bar{\rho} - a_0}{\rho \bar{\rho} + a_0}, k \right), \frac{1}{\omega_1} \operatorname{cn}^{-1} \left(\frac{\rho \bar{\rho} - a_0}{\rho \bar{\rho} + a_0}, k \right) \right),$$
(4.3)

and

$$\phi(\xi) = \pm \left(\frac{\rho\bar{\rho}(1-\operatorname{cn}(\omega_{2}\xi,k))}{(1+\operatorname{cn}(\omega_{2}\xi,k))}\right)^{\frac{1}{2}}, \xi \in \left(-\frac{1}{\omega_{2}}\operatorname{cn}^{-1}\left(\frac{\rho\bar{\rho}-a_{0}}{\rho\bar{\rho}+a_{0}},k\right), \frac{1}{\omega_{2}}\operatorname{cn}^{-1}\left(\frac{\rho\bar{\rho}-a_{0}}{\rho\bar{\rho}+a_{0}},k\right)\right),$$
where $k^{2} = -\frac{(\rho-\bar{\rho})^{2}}{(\bar{\rho}-\bar{\rho})^{2}}, \omega_{1} = 2\sqrt{\frac{1}{\pi}|\alpha_{4}|\rho\bar{\rho}}, \omega_{2} = 2\sqrt{\frac{1}{\pi}|\alpha_{4}|\rho\bar{\rho}}.$
(4.4)

where $k^2 = -\frac{(\rho-\rho)}{4\rho\bar{\rho}}, \omega_1 = 2\sqrt{\frac{1}{3}}|\alpha_4|\rho\bar{\rho}, \omega_2 = 2\sqrt{\frac{1}{2}}|\alpha_4|\rho\bar{\rho}.$ Notice that the level curves defined by $H_1(\phi, y) = h$ and $H_2(\phi, y) = h, h \in (0, h_1)$

Notice that the level curves defined by $H_1(\phi, y) = h$ and $H_2(\phi, y) = h, h \in (0, h_1)$ contain a global family of closed orbits enclosing five equilibrium points. When $h \to 0$, this family of periodic orbits attend to the curve quadrangle defined by $H_j(\phi, y) = 0, j = 1, 2$. When $0 < h \ll 0$ These periodic orbits give rise to a family of periodic peakons. As a limit solution, (4.3) and (4.4) are also give rise two periodic peakon solutions of system (1.5).

4.2. The case of Figure.3 (b).

For $\beta = 1, 2$, the level curves defined by $H_1(\phi, y) = 0$ and $H_2(\phi, y) = 0$ contain two heteroclinic orbits, connecting two equilibrium points $(\pm \phi_1, 0)$ and enclosing the origin, and two curve triangles enclosing the equilibrium points $(\pm \phi_2, 0)$. Now, we have $y^2 = \frac{|\alpha_4|}{3}(\phi_1^2 - \phi^2)^2$, for $\beta = 1$ and $y^2 = \frac{|\alpha_4|}{2}(\phi_1^2 - \phi^2)^2$, for $\beta = 2$. Hence, the two heteroclinic orbits has the parametric representations (kink and anti-kink wave solutions):

$$\phi(\xi) = \pm \phi_1 \tanh(\omega_j \xi), \quad j = 3, 4,$$
(4.5)

where $\omega_3 = \sqrt{\frac{1}{3}|\alpha_4|}\phi_1, \omega_4 = \sqrt{\frac{1}{2}|\alpha_4|}\phi_1$. The two segments of the left curve triangle have the parametric representations:

$$\phi(\xi) = \pm \frac{(-\phi_1)(e^{\omega_j \xi} + m_0 e^{-\omega_j \xi})}{e^{\omega_j \xi} - m_0 e^{-\omega_j \xi}}, \quad j = 3, 4,$$
(4.6)

where $m_0 = \frac{\sqrt{a_0} + \phi_1}{\sqrt{a_0} - \phi_1}$.

As a limit solution of a family of periodic orbits defined by $H_j(\phi, y) = h, h \in (0, h_2)$ when $h \to 0$, the parametric representations (4.6) of two curve triangles give rise to a peakon and an anti-peakon solutions of system (1.5).

4.3. The case of Figure.3 (c).

(i) For $\beta = 1, 2$, the level curves defined by $H_j(\phi, y) = 0, j = 1, 2$, are two straight lines $\phi = \sqrt{a_0}$ and two arches enclosing the centers $(\pm \phi_2, 0)$, respectively. In these cases, we have $y^2 = \frac{1}{3} |\alpha_4| (\phi - r_m^2) (r_1^2 + \phi^2)$ and $y^2 = \frac{1}{2} |\alpha_4| (\phi - r_m^2) (r_1^2 + \phi^2)$. Thus,

we obtain the following parametric representations of periodic peakon solutions of system (1.5):

$$\phi(\xi) = \frac{\pm r_m}{\operatorname{cn}(\omega_j \xi, k)}, \quad \xi \in \left(-\frac{1}{\omega_j} \operatorname{cn}^{-1}\left(\frac{r_m}{\sqrt{a_0}}, k\right), \frac{1}{\omega_j} \operatorname{cn}^{-1}\left(\frac{r_m}{\sqrt{a_0}}, k\right)\right), \quad j = 5, 6,$$
(4.7)
where $k^2 = \frac{r_1^2}{r_2^2 + r_2^2}, \omega_5 = \sqrt{\frac{1}{3}|\alpha_4|(r_1^2 + r_m^2)}, \omega_6 = \sqrt{\frac{1}{2}|\alpha_4|(r_1^2 + r_m^2)}.$

(ii) For $\beta = 1$, the level curves defined by $H_1(\phi, y) = h_1$ contain two heteroclinic orbits connecting the equilibrium points $(\pm \phi_1, 0)$ and enclosing the center (0, 0), and two open curves passing through the ϕ -axis at $(\pm r_1, 0)$ with $\phi_2 < \sqrt{a_0} < r_1$. For the heteroclinic orbits, we have $2\sqrt{\frac{|\alpha_4|}{3}}\xi = \int_0^u \frac{(a_0-u)du}{(\phi_1^2-u)\sqrt{(r_1^2-u)(a_0-u)u}}$. Thus, we obtain the following parametric representations of kink and anti-kink wave solutions of system (1.5):

$$\phi(\chi) = \pm \sqrt{a_0} \operatorname{sn}(\chi, k), \quad \chi \in \left(-\operatorname{sn}^{-1}\left(\frac{\phi_1}{\sqrt{a_0}}, k\right), \operatorname{sn}^{-1}\left(\frac{\phi_1}{\sqrt{a_0}}, k\right)\right),$$

$$\xi(\chi) = \sqrt{\frac{3}{r_1^2 |\alpha_4|}} \left[\chi + \left(1 - \frac{a_0}{\phi_1^2}\right) \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \hat{\alpha}_3^2, k)\right],$$
(4.8)

where $k^2 = \frac{a_0}{r_1^2}, \hat{\alpha}_3^2 = \frac{a_0}{\phi_1^2}.$

For $\beta = 2$, the level curves defined by $H_2(\phi, y) = h_1$ contain two heteroclinic orbits enclosing the center (0, 0) and connecting the equilibrium points $(\pm \phi_1, 0)$. Now, we have $y^2 = \frac{\frac{1}{2}|\alpha_4|(\phi_1^2 - \phi^2)^2(\phi^2 - \rho^2)(\phi^2 - \bar{\rho}^2)}{(a_0 - \phi^2)}$, where ρ is a complex number. By using the integral $2\sqrt{\frac{|\alpha_4|}{2}}\xi = \int_0^u \frac{(a_0 - u)du}{(\phi_1^2 - u)\sqrt{u[(u - b_1)^2 + a_1^2]}}$, where $a_1^2 = -\frac{1}{4}(\rho^2 - \bar{\rho}^2)^2$, $b_1 = \frac{1}{2}(\rho^2 + \bar{\rho}^2)$, we obtain the following parametric representations of kink and anti-kink wave solutions of system (1.5):

$$\begin{split} \phi(\chi) &= \pm \left(\frac{A_1(1-\operatorname{cn}(\chi,k))}{(1+\operatorname{cn}(\chi,k))}\right)^{\frac{1}{2}}, \quad \chi \in \left(-\operatorname{cn}^{-1}\left(\frac{A_1-\phi_1^2}{A_1+\phi_1^2},k\right), \operatorname{cn}^{-1}\left(\frac{A_1-\phi_1^2}{A_1+\phi_1^2},k\right)\right), \\ \xi(\chi) &= \frac{1}{\sqrt{2|\alpha_4|(a_1^2+b_1^2)}} \left[\left(\frac{a_0+A_1}{\phi_1^2+A_1}\right)\chi - \frac{(a_0-\phi_1^2)}{2\phi_1^2\hat{\alpha}}\Pi\left(\operatorname{arcsin}(\operatorname{cn}(\chi,k)),\frac{\hat{\alpha}^2}{\hat{\alpha}^2-1},k\right) + \frac{(a_0-\phi_1^2)\hat{\alpha}}{(\phi_1^2+A_1)(1+\hat{\alpha})}f_1\right], \end{split}$$
(4.9)

where $A_1^2 = b_1^2 + a_1^2$, $k^2 = \frac{A_1+b_1}{2A_1}$, $\hat{\alpha} = \frac{\phi_1^2+A_1}{\phi_1^2-A_1}$, f_1 can be seen in [Byrd & Fridman, 1971](361.54).

4.4. The case of Figure.3 (d) and (e).

For $\beta = 1, 2$, the level curves defined by $H_j(\phi, y) = h_1, j = 1, 2$ contain two heteroclinic orbits, which have the same parametric representations as (4.8) and (4.9).

4.5. The case of Figure.3 (f).

For $\beta = 1, 2$, the level curves defined by $H_j(\phi, y) = 0, j = 1, 2$, are two straight lines $\phi = \pm \sqrt{a_0}$ and a global closed orbit enclosing (0,0) and $(\pm \phi_1, 0)$, which contacts to two straight lines at $(\pm \sqrt{a_0}, 0)$. In these cases, we have $y^2 = \frac{1}{3} |\alpha_4| (r_1^2 - \phi^2) (a_0 - \phi^2)$ and $y^2 = \frac{1}{2} |\alpha_4| (r_1^2 - \phi) (a_0 - \phi^2)$. Thus, we obtain the following parametric representations of smooth periodic wave solution of system (1.5):

$$\phi(\xi) = \sqrt{a_0} \operatorname{sn}(\omega_j \xi, k), \quad j = 7, 8,$$
(4.10)

where $k^2 = \frac{a_0}{r_1^2}, \omega_7 = \sqrt{\frac{1}{3}|\alpha_4|r_1^2}, \omega_8 = \sqrt{\frac{1}{2}|\alpha_4|r_1^2}.$

4.6. The case of Figure.3 (g).

For $\beta = 1, 2$, the level curves defined by $H_j(\phi, y) = 0, j = 1, 2$ are two straight lines $\phi = \pm \sqrt{a_0}$ and a global closed orbit enclosing three centers $(0,0), (\pm \phi_1,0)$, which passes through two straight lines. In these cases, we have $y^2 = \frac{1}{3}|\alpha_4|(r_1^2 - \phi^2)(r_2^2 - \phi^2)$ and $y^2 = \frac{1}{2}|\alpha_4|(r_1^2 - \phi)(r_2^2 - \phi^2)$. Thus, we obtain the following parametric representations of smooth periodic wave solutions of system (1.5):

 $\phi(\xi) = r_2 \operatorname{sn}(\omega_j \xi, k), \quad j = 9, 10, \tag{4.11}$

where $k^2 = \frac{r_2^2}{r_1^2}, \omega_9 = \sqrt{\frac{1}{3}|\alpha_4|r_1^2}, \omega_{10} = \sqrt{\frac{1}{2}|\alpha_4|r_1^2}.$

5. Exact parametric representations of solutions of system (1.5) when $\beta = -3, -4$ in Figure.4

In this section, we discuss possible parametric representations of the level curves defined by $H_j(\phi, y) = h, j = -3, -4$, in (2.4) and (2.5) for the case 3 in section 2. Notice that $H_j(\phi, y) = h, j = -3, -4$, give rise to

$$y^{2} = h\phi^{6} - 3(ha_{0} + \alpha_{4})\phi^{4} + 3\left(ha_{0}^{2} + \alpha_{4}a_{0} - \frac{1}{2}\alpha_{2}\right)\phi^{2} - C_{0} = \tilde{F}_{6}(\phi), \quad (5.1)$$

where $C_0 = ha_0^3 + \alpha_0 - \frac{1}{2}\alpha_2 a_0 + \alpha_4 a_0^2$. and

$$y^{2} = h\phi^{8} - 4ha_{0}\phi^{6} + 2(3ha_{0}^{2} - \alpha_{4})\phi^{4} - 4\left(ha_{0}^{3} - \frac{1}{3}\alpha_{4}a_{0} + \frac{1}{3}\alpha_{2}\right)\phi^{2} + D_{0} = \tilde{F}_{8}(\phi),$$
(5.2)

where $D_0 = ha_0^4 - \alpha_0 + \frac{1}{3}\alpha_2 a_0 - \frac{1}{3}\alpha_4 a_0^2$.

In this case, we have $\alpha_0 > 0, \alpha_2 < 0, \alpha_4 > 0$. So that, for $\beta = -3$, we have $h_0 = -\frac{2a_0 - a_0\alpha_2 + 2a_0^2\alpha_4}{2a_0^3}$,

$$h_1 = -\frac{2\alpha_4^2[(3\alpha_2^2 + 4a_0\alpha_2\alpha_4 + 4a_0^2\alpha_4^2 - 8\alpha_0\alpha_4) + (3\alpha_2 + 6a_0\alpha_4)\sqrt{\Delta}]}{(\sqrt{\Delta} + (\alpha_2 + 2a_0\alpha_4))^3}$$
$$h_2 = \frac{2\alpha_4^2[(3\alpha_2^2 + 4a_0\alpha_2\alpha_4 + 4a_0^2\alpha_4^2 - 8\alpha_0\alpha_4) - (3\alpha_2 + 6a_0\alpha_4)\sqrt{\Delta}]}{(\sqrt{\Delta} - (\alpha_2 + 2a_0\alpha_4))^3}.$$

For $\beta = -4$, we have $h_0 = \frac{3\alpha_0 - a_0\alpha_2 + a_0^2\alpha_4}{2a_0^4}$,

$$h_1 = \frac{16\alpha_4^3[(a_0^2\alpha_4^2 + (a_0\alpha_2 - 3\alpha_0)\alpha_4 + \alpha_2^2) + (\alpha_2 + 2a_0\alpha_4)\sqrt{\Delta}]}{3(\alpha_2 + 2a_0\alpha_4 + \sqrt{\Delta})^4}$$

$$h_2 = \frac{16\alpha_4^3[(a_0^2\alpha_4^2 + (a_0\alpha_2 - 3\alpha_0)\alpha_4 + \alpha_2^2) - (\alpha_2 + 2a_0\alpha_4)\sqrt{\Delta}]}{3(\alpha_2 + 2a_0\alpha_4 - \sqrt{\Delta})^4}$$

By using the first equation of system (1.5), we know that $\xi = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\tilde{F}_6(\phi)}}$ and $\xi = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\tilde{F}_8(\phi)}}$. Thus, we can calculate the parametric representations for all bounded orbits given by Figure.4 and Figure.5 for $\beta = -3, -4$.

5.1. The case of Figure 4 (a).

(i) For $\beta = -3$, we consider the case $h_2 < 0$. The level curves defined by $H_{-3}(\phi, y) = h, h \in (-\infty, h_1)$ are a global family of periodic orbits of system (1.5) enclosing three equilibrium points. Now, we have from (5.1) that $\sqrt{|h|}\xi = \int_0^{\phi} \frac{d\phi}{\sqrt{(r_1^2 - \phi^2)(\phi^2 - \bar{\rho}^2))}}$. It gives rise to the following parametric representation of the periodic solutions of system (1.5):

$$\phi(\xi) = \left(\frac{r_1^2 B_1^2 (1 - \operatorname{cn}(\Omega_1 \xi, k))}{(A_1 + B_1) + (A_1 - B_1) \operatorname{cn}(\Omega_1 \xi, k)}\right)^{\frac{1}{2}},\tag{5.3}$$

where $A_1 = (r_1^2 - b_1)^2 + a_1^2$, $B_1^2 = a_1^2 + b_1^2 = \rho^2 \bar{\rho}^2$, $a_1^2 = -\frac{1}{4}(\rho^2 - \bar{\rho}^2)^2$, $b_1 = \frac{1}{2}(\rho^2 + \bar{\rho}^2)$, $k^2 = \frac{r_1^4 - (A_1 - B_1)^2}{4A_1B_1}$, $\Omega_1 = 2\sqrt{|h|A_1B_1}$. The level curves defined by $H_{-3}(\phi, y) = h_1$ contain two homoclinic orbits to the

The level curves defined by $H_{-3}(\phi, y) = h_1$ contain two homoclinic orbits to the equilibrium points $(\pm \phi_1, 0)$, respectively, and two heteroclinic orbits connecting two points $(\pm \phi_1, 0)$. Corresponding to the above heteroclinic orbit, we have $\sqrt{|h_1|}\xi = \int_0^{\phi} \frac{d\phi}{(\phi_1^2 - \phi^2)\sqrt{\phi_M^2 - \phi^2}}$. Thus, the two heteroclinic orbits have the following parametric representations:

$$\phi(\xi) = \pm \left(\phi_1^2 - \frac{2\phi_1^2(\phi_M^2 - \phi_1^2)}{\phi_M^2 \cosh(\omega_1\xi) - (2\phi_1^2 - \phi_M^2)}\right)^{\frac{1}{2}},\tag{5.4}$$

where $\omega_1 = 2\sqrt{|h_1|\phi_1(\phi_M^2 - \phi_1^2)}$. Corresponding to the right homoclinic orbit, we have $\sqrt{|h_1|}\xi = \int_{\phi}^{\phi_M} \frac{d\phi}{(\phi - \phi_1^2)\sqrt{\phi_M^2 - \phi^2}}$. Thus, the two homoclinic orbits have the following parametric representations:

$$\phi(\xi) = \pm \left(\phi_1^2 + \frac{2\phi_1^2(\phi_M^2 - \phi_1^2)}{\phi_M^2 \cosh(\omega_1\xi) + (2\phi_1^2 - \phi_M^2)}\right)^{\frac{1}{2}}.$$
(5.5)

The level curves defined by $H_{-3}(\phi, y) = h, h \in (h_1, h_0)$ contain three families of periodic orbits of system (1.5) enclosing the equilibrium points $(\pm \phi_2, 0)$ and (0,0), respectively. For the right family of periodic orbits, we have $\sqrt{|h|\xi} = \int_{r_2}^{\phi} \frac{d\phi}{\sqrt{(r_1^2 - \phi^2)(\phi^2 - r_2^2)(\phi^2 - r_3^2)}}$. Hence, we obtain the following parametric representations of two families of periodic orbits of system (1.5):

$$\phi(\xi) = \pm \left(r_3^2 + \frac{r_2^2 - r_3^2}{1 - \hat{\alpha}_1^2 \mathrm{sn}^2(\Omega_2 \xi, k)} \right)^{\frac{1}{2}},\tag{5.6}$$

where $\hat{\alpha}_1^2 = \frac{r_1^2 - r_2^2}{r_1^2 - r_3^2}, k^2 = \frac{\hat{\alpha}_1^2 r_3^2}{r_2^2}, \Omega_2 = r_2 \sqrt{|h|(r_1^2 - r_3^2)}$. For the mid family of periodic orbits, we have $\sqrt{|h|}\xi = \int_0^\phi \frac{d\phi}{\sqrt{(r_1^2 - \phi^2)(r_2^2 - \phi^2)(r_3^2 - \phi^2)}}$. We obtain the following parametric representations of the family of periodic orbits of system (1.5):

$$\phi(\xi) = \left(r_1^2 - \frac{r_1^2}{1 - \hat{\alpha}_2^2 \mathrm{sn}^2(\Omega_2 \xi, k)}\right)^{\frac{1}{2}},\tag{5.7}$$

where $\hat{\alpha}_2^2 = -\frac{r_3^2}{r_1^2 - r_3^2}, k^2 = \frac{-\hat{\alpha}_2^2(r_1^2 - r_2^2)}{r_2^2}.$

The level curves defined by $H_{-3}(\phi, y) = h, h \in (h_0, h_2)$ contain two families of periodic orbits of system (1.5) enclosing the equilibrium points $(\pm \phi_2, 0)$, respectively. In this case, we have $\sqrt{|h|}\xi = \int_{r_2}^{\phi} \frac{d\phi}{\sqrt{(r_1^2 - \phi^2)(\phi^2 - r_2^2)(\phi^2 + r_3^2)}}$. Thus, we obtain the following parametric representations of two families of periodic orbits of system (1.5):

$$\phi(\xi) = \pm \left(\frac{r_2^2}{1 - \hat{\alpha}_3^2 \mathrm{sn}^2(\Omega_3 \xi, k)}\right)^{\frac{1}{2}},\tag{5.8}$$

where $\hat{\alpha}_3^2 = \frac{r_1^2 - r_2^2}{r_1^2}, k^2 = \frac{\hat{\alpha}_3^2 r_3^2}{r_2^2 + r_3^2}, \Omega_3 = r_1 \sqrt{|h|(r_2^2 + r_3^2)}.$ (ii) For $\beta = -4$, the level curves defined by $H_{-4}(\phi, y) = h_1$ contain two homo-

(ii) For $\beta = -4$, the level curves defined by $H_{-4}(\phi, y) = h_1$ contain two homoclinic orbits to the equilibrium points $(\pm \phi_1, 0)$, respectively, and two heteroclinic orbits connecting two points $(\pm \phi_1, 0)$. Corresponding to the above heteroclinic orbit, we have $\sqrt{h_1}\xi = \int_0^{\phi} \frac{d\phi}{(\phi_1^2 - \phi^2)\sqrt{(r_1^2 - \phi^2)(\phi_M^2 - \phi^2)}}$. Thus, the two heteroclinic orbits have the following parametric representations:

$$\begin{aligned} \phi(\chi) &= \pm \phi_M \operatorname{sn}(\chi, k), \quad \chi \in \left(-\operatorname{sn}^{-1}\left(\frac{\phi_1}{\phi_M}, k\right), \operatorname{sn}^{-1}\left(\frac{\phi_1}{\phi_M}, k\right) \right), \\ \xi(\chi) &= \frac{1}{r_1 \phi_1^2 \sqrt{h_1}} \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \hat{\alpha}_4^2, k), \end{aligned}$$
(5.9)

where $\hat{\alpha}_4 = \frac{\phi_M}{\phi_1}, k = \frac{\phi_M}{r_1}$. Corresponding to the right homoclinic orbit, we have $\sqrt{h_1}\xi = \int_{\phi}^{\phi_M} \frac{d\phi}{(\phi-\phi_1^2)\sqrt{(r_1^2-\phi^2)(\phi_M^2-\phi^2)}}$. Thus, the two homoclinic orbits have the following parametric representations:

$$\begin{split} \phi(\chi) &= \pm \frac{\phi_M \operatorname{cn}(\chi, k)}{\operatorname{dn}(\chi, k)}, \quad \chi \in \left(-\operatorname{sn}^{-1}\left(\sqrt{\frac{r_1^2(\phi_M^2 - \phi_1^2)}{\phi_M^2(r_1^2 - \phi_1^2)}}, k \right), \operatorname{sn}^{-1}\left(\sqrt{\frac{r_1^2(\phi_M^2 - \phi_1^2)}{\phi_M^2(r_1^2 - \phi_1^2)}}, k \right) \right), \\ \xi(\chi) &= \frac{1}{r_1(r_1^2 - \phi_1^2)\sqrt{h_1}} \left[\chi + \frac{r_1^2 - \phi_M^2}{\phi_M^2 - \phi_1^2} \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \hat{\alpha}_5^2, k) \right], \\ \text{where } \hat{\alpha}_5^2 &= \frac{k^2(r_1^2 - \phi_1^2)}{\phi_M^2 - \phi_1^2}, k = \frac{\phi_M}{r_1}. \end{split}$$
(5.10)

5.2. The case of Figure.4 (b).

(i) For $\beta = -3$, the level curves defined by $H_{-3}(\phi, y) = h, h \in (-\infty, h_1)$ are a global family of closed orbits which contacts to two singular straight lines at $(\pm \sqrt{a_0}, 0)$. It has similar parametric representation as (5.3), where we use a_0 instead of r_1^2 .

The level curves defined by $H_{-3}(\phi, y) = h_1$ contain two homoclinic orbits to the equilibrium points $(\pm \phi_1, 0)$, respectively, and two heteroclinic orbits connecting two points $(\pm \phi_1, 0)$. They have similar parametric representations as (5.4) and (5.5), where we use a_0 instead of ϕ_M^2 .

The level curves defined by $H_{-3}(\phi, y) = h, h \in (h_1, h_0)$ contain three families of closed orbits of system (1.5) enclosing the equilibrium point (0,0) and contact to the equilibrium points $(\pm \sqrt{a_0}, 0)$, respectively. They have similar parametric representations as (5.6) and (5.7), where we use a_0 instead of r_1^2 .

The level curves defined by $H_{-3}(\phi, y) = h, h \in (h_0, 0)$ contain two families of closed orbits of system (1.5) contacting to the points $(\pm \sqrt{a_0}, 0)$, respectively. They have similar parametric representations as (5.8), where we use a_0 instead of r_1^2 .

The level curves defined by $H_{-3}(\phi, y) = h, h \in (0, \infty)$ contain two families of closed orbits of system (1.5) contacting to the points $(\pm \sqrt{a_0}, 0)$, respectively, and two open curves passing through ϕ -axis at the points $(\pm r_1, 0)$. In this case, we have $\sqrt{h\xi} = \int_{r_3}^{\phi} \frac{d\phi}{\sqrt{(r_1^2 - \phi^2)(a_0 - \phi^2)(\phi^2 - r_3^2)}}$. Therefore, we have the parametric representations:

$$\phi(\xi) = \pm \left(\frac{r_3^2}{1 - \hat{\alpha}_6^2 \mathrm{sn}^2(\Omega_4 \xi, k)}\right)^{\frac{1}{2}},\tag{5.11}$$

where $k^2 = \frac{(a_0 - r_3^2)r_1^2}{a_0(r_1^2 - r_3^2)}, \hat{\alpha}_6^2 = \frac{a_0 - r_3^2}{a_0}, \Omega_4 = \sqrt{ha_0(r_1^2 - r_3^2)}.$ (ii) For $\beta = -4$, the level curves defined by $H_{-4}(\phi, y) = h_1$ contain two ho-

(ii) For $\beta = -4$, the level curves defined by $H_{-4}(\phi, y) = h_1$ contain two homoclinic orbits to the equilibrium points $(\pm \phi_1, 0)$ and contacting to two singular straight lines $\phi = \pm \sqrt{a_0}$, respectively, and two heteroclinic orbits connecting two points $(\pm \phi_1, 0)$. They have similar parametric representations as (5.9) and (5.10), where we use a_0 instead of ϕ_M^2 .

5.3. The case of Figure 4 (c).

We see from Figure.4 (c) that in the straight lines $\phi = \pm \sqrt{a_0}$, there exist two nodes of system (2.1). For singular system (1.5), when a phase point passes through the above two straight lines the vector field defined by system (1.5) changes to the inverse direction defined by system (2.1). The following Figure.7 (a)-(f) give the changes of the level curves defined by system (1.5).



Figure. 7 The changes of the level curves defined by $H_{-3}(\phi, y) = h$ for in Figure.4 (c)

(i) For $\beta = -3$, corresponding the level curves defined by $H_{-3}(\phi, y) = h, h \in (-\infty, h_2)$, the parametric representations of orbits of system (1.5) are the same as (5.3)-(5.8).

Along the two homoclinic orbits to the equilibrium points $(\pm \phi_2, 0)$ of system (1.5) defined by $H_{-3}(\phi, y) = h_2$, We have $\sqrt{h}\xi = \int_{\phi_M}^{\phi} \frac{d\phi}{(\phi_2^2 - \phi_2)\sqrt{\phi^2 - \phi_M^2}}$. Therefore,

we obtain the following parametric representations:

$$\phi(\xi) = \pm \left(\phi_2^2 - \frac{2\phi_2^2(\phi_2^2 - \phi_M^2)}{\phi_M^2 \cosh(\omega_2\xi) + (2\phi_2^2 - \phi_M^2)}\right)^{\frac{1}{2}},\tag{5.12}$$

where $\omega_2 = 2\phi_2 \sqrt{h(\phi_2^2 - \phi_M^2)}$. (ii) For $\beta = -4$, The level curves defined by $H_{-4}(\phi, y) = h_2$ are two homoclinic orbits to the equilibrium points $(\pm \phi_2, 0)$. we have $\sqrt{h}\xi = \int_{\phi_M}^{\phi} \frac{d\phi}{(\phi_2^2 - \phi^2)\sqrt{(\phi^2 - \phi_M^2)(\phi^2 + r_1^2)}}$ Hence, we obtain the following parametric representations:

$$\phi(\chi) = \pm \frac{\phi_M}{\operatorname{cn}(\chi,k)}, \quad \chi \in \left(-\operatorname{cn}^{-1}\left(\frac{\phi_M}{\phi_2},k\right), \operatorname{cn}^{-1}\left(\frac{\phi_M}{\phi_2},k\right)\right),$$

$$\xi(\chi) = \frac{1}{\phi_2^2 \sqrt{h(\phi_M^2 + r_1^2)}} \left[\chi + \frac{\phi_M^2}{\phi_2^2 - \phi_M^2} \Pi(\operatorname{arccos}(\operatorname{cn}(\chi,k)), \hat{\alpha}_7^2, k)\right],$$
(5.13)

where $k^2 = \frac{r_1^2}{\phi_M^2 + r_1^2}, \hat{\alpha}_7^2 = \frac{\phi_2^2}{\phi_2^2 - \phi_M^2}.$

5.4. The case of Figure.4 (d).

(i) For $\beta = -3$, the level curves defined by $H_{-3}(\phi, y) = h, h \in (-\infty, h_0)$ contain three families of closed orbits of system (1.5) enclosing the equilibrium point (0,0)and contact to the equilibrium points $(\pm \sqrt{a_0}, 0)$, respectively. They have similar parametric representations as (5.6) and (5.7), where we use a_0 instead of r_2^2 .

The level curves defined by $H_{-3}(\phi, y) = h, h \in (h_0, 0)$ contain two families of closed orbits of system (1.5) contacting to the points $(\pm \sqrt{a_0}, 0)$, respectively. They have similar parametric representations as (5.8), where we use a_0 instead of r_2^2 .

The level curves defined by $H_{-3}(\phi, y) = h, h \in (0, h_2)$ contain two families of closed orbits of system (1.5) contacting to the points $(\pm \sqrt{a_0}, 0)$, respectively and two open curves passing through ϕ -axis at the points $(\pm r_1, 0)$. Two families of closed orbits of system (1.5) have the similar parametric representations as (5.11), where we use a_0 instead of r_3^2 and use r_2^2 instead of a_0 in (5.11).

The level curves defined by $H_{-3}(\phi, y) = h_2$ are two homoclinic orbits to the equilibrium points $(\pm \phi_2, 0)$ and contact to two straight lines $\phi = \pm \sqrt{a_0}$. They have the similar parametric representations as (5.12), where we use a_0 instead of ϕ_M^2 .

(ii) For $\beta = -4$, the level curves defined by $H_{-4}(\phi, y) = h_2$ are two homoclinic orbits to the equilibrium points $(\pm \phi_2, 0)$ and contact to two straight lines $\phi = \pm \sqrt{a_0}$. They have the similar parametric representations as (5.13), where we use a_0 instead of ϕ_M^2 .

5.5. The case of Figure 4 (e).

(i) For $\beta = -3$, the level curves defined by $H_{-3}(\phi, y) = h, h \in (-\infty, h_0)$ are a family of periodic orbits of system (1.5) enclosing the origin (0,0), which has the same parametric representation as (5.3).

The level curves defined by $H_{-3}(\phi, y) = h, h \in (h_1, 0)$ are two families of periodic orbits of system (1.5) enclosing two equilibrium points $(\pm \phi_1, 0)$, which have the same parametric representations as (5.8).

The level curves defined by $H_{-3}(\phi, y) = h, h \in (0, h_2)$ are two families of periodic orbits of system (1.5) enclosing two equilibrium points $(\pm \phi_1, 0)$ and two open curves passing through ϕ -axis at the points ($\pm r_1, 0$). Two families of periodic orbits have the similar parametric representations as (5.11), where we use r_2^2 instead of a_0 .

The level curves defined by $H_{-3}(\phi, y) = h_2$ are two homoclinic orbits to the equilibrium points $(\pm \phi_2, 0)$, which have the same parametric representations as (5.12).

(ii) For $\beta = -4$, the level curves defined by $H_{-4}(\phi, y) = h_2$ are two homoclinic orbits to the equilibrium points $(\pm \phi_2, 0)$, which have the same parametric representations as (5.13).

6. Exact parametric representations of solutions of system (1.5) when $\beta = -3, 4$ in Figure.5

In this section, we discuss possible parametric representations of the level curves defined by $H(\phi, y) = h$ in (2.4) and (2.5) for the case 4 in section 2.

6.1. The case of Figure.5 (a).

(i) For $\beta = -3$, we assume that $h_2 < 0$. In the two straight lines $\phi = \pm \sqrt{a_0}$ there exist four node points of system (2.1). The level curves defined by $H_{-3}(\phi, y) = h, h \in (-\infty, h_2)$ contain a global family of closed orbits of system (1.5) enclosing three equilibrium points and pass though two straight lines $\phi = \pm \sqrt{a_0}$, respectively (see Figure 7 (a)). It has the same parametric representation as (5.3).

The level curves defined by $H_{-3}(\phi, y) = h_2$ contain two homoclinic orbits to the equilibrium points $(\pm \phi_2, 0)$, respectively, and two heteroclinic orbits connecting two points $(\pm \phi_2, 0)$, which have the similar parametric representations as (5.4) and (5.5), where we use ϕ_2^2 instead of ϕ_1^2 .

The level curves defined by $H_{-3}(\phi, y) = h, h \in (h_2, 0)$ contain three families of periodic orbits of system (1.5) enclosing the equilibrium points $(\pm \phi_1, 0)$ and (0, 0), which have the same parametric representations as (5.6) and (5.7).

The level curves defined by $H_{-3}(\phi, y) = h, h \in (0, h_0)$ contain a family of closed orbits of system (1.5) enclosing three equilibrium points and two open curves passing through the ϕ -axis at the points ($\pm r_1, 0$) and connecting two node points, respectively. In this case, for the periodic family, we have $\sqrt{h\xi} = \int_0^{\phi} \frac{d\phi}{\sqrt{(r_1^2 - \phi^2)(r_2^2 - \phi^2)(\phi^2 + r_3^2)}}$ Thus, we have the parametric representation of the periodic solution family as follows:

$$\phi(\xi) = \left(\frac{r_3^2 \check{\alpha}_1^2 \mathrm{sn}^2(\check{\Omega}_1 \xi, k)}{1 - \check{\alpha}_1^2 \mathrm{sn}^2(\check{\Omega}_1 \xi, k)}\right)^{\frac{1}{2}},\tag{6.1}$$

where $\check{\alpha}_{1}^{2} = \frac{r_{2}^{2}}{r_{2}^{2} + r_{3}^{2}}, k^{2} = \frac{\check{\alpha}_{1}^{2}(r_{1}^{2} + r_{3}^{2})}{r_{1}^{2}}, \check{\Omega}_{1} = r_{1}\sqrt{h(r_{2}^{2} + r_{3}^{2})}.$ The level curves defined by $H_{-3}(\phi, y) = h_{0}$ contain two homoclinic orbits of

The level curves defined by $H_{-3}(\phi, y) = h_0$ contain two homoclinic orbits of system (1.5) to the origin and two open curves passing through the ϕ -axis at the points $(\pm r_1, 0)$ and connecting two node points, respectively. In this case, for the right homoclinic orbit, we have $\sqrt{h_0}\xi = \int_{\phi}^{\phi_M} \frac{d\phi}{\phi\sqrt{(\phi_M^2 - \phi^2)(r_1^2 - \phi^2)}}$. Therefore, we have the parametric representations of two homoclinic orbits of system (1.5) as follows:

$$\phi(\xi) = \pm \left(\frac{2r_1^2\phi_M^2}{(r_1^2 - \phi_M^2)\cosh(\tilde{\omega}_1\xi) + (r_1^2 + \phi_M^2)}\right)^{\frac{1}{2}},\tag{6.2}$$

where $\check{\omega}_1 = 2r_1\phi_M\sqrt{h_0}$.

The level curves defined by $H_{-3}(\phi, y) = h, h \in (h_0, h_1)$ contain two families of closed orbits of system (1.5) enclosing the equilibrium points $(\pm \phi_1, 0)$ and two

open curves passing through the ϕ -axis at the points $(\pm r_1, 0)$ and connecting two node points, respectively. In this case, for the right periodic family, we have $\sqrt{h}\xi = \int_{r_3}^{\phi} \frac{d\phi}{\sqrt{(r_1^2 - \phi^2)(r_2^2 - \phi^2)(\phi^2 - r_3^2)}}$. Thus, the parametric representations of two periodic solution families are similar to (5.11), where we use r_2^2 instead of a_0 .

(ii) For $\beta = -4$, we consider the case $h_1 < h_0 < h_2 < 0$. The level curves defined by $H_{-4}(\phi, y) = h_0 < 0$, contain two homoclinic orbits to the origin and two closed orbits passing through two straight lines $\phi = \pm \sqrt{a_0}$, respectively. For the right homoclinic orbit, we have $\sqrt{|h_0|}\xi = \int_{\phi}^{\phi_M} \frac{d\phi}{\phi\sqrt{(r_1^2 - \phi^2)(r_2^2 - \phi^2)(\phi_M^2 - \phi^2)}}$. Thus, we obtain the parametric representations of two homoclinic solutions as follows:

$$\begin{split} \phi(\chi) &= \pm \left(r_2^2 - \frac{r_2^2 - \phi_M^2}{\operatorname{cn}^2(\chi, k)} \right)^{\frac{1}{2}}, \quad \chi \in \left(-\operatorname{cn}^{-1} \left(\sqrt{1 - \frac{\phi_M^2}{r_2^2}}, k \right), \operatorname{cn}^{-1} \left(\sqrt{1 - \frac{\phi_M^2}{r_2^2}}, k \right) \right), \\ \xi(\chi) &= \frac{1}{r_2^2 \sqrt{|h_0|(r_1^2 - \phi_M^2)}} \left[\chi + \frac{r_2^2 - \phi_M^2}{\phi_M^2} \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \check{\alpha}_2^2, k) \right], \end{split}$$

$$(6.3)$$

where $k^2 = \frac{r_1^2 - r_2^2}{r_1^2 - \phi_M^2}, \check{\alpha}_2^2 = \frac{r_2^2}{\phi_M^2}.$

For the right periodic orbit, we have $\sqrt{|h_0|}\xi = \int_{r_2}^{\phi} \frac{d\phi}{\phi\sqrt{(r_1^2 - \phi^2)(\phi^2 - r_2^2)(\phi^2 - \phi_M^2)}}$ Hence, we have the following parametric representations of two periodic solutions of system (1.5):

$$\begin{aligned} \phi(\chi) &= \pm \left(\phi_M^2 + \frac{r_2^2 - \phi_M^2}{\mathrm{dn}^2(\chi, k)}\right)^{\frac{1}{2}}, \\ \xi(\chi) &= \frac{1}{\phi_M^2 \sqrt{|h_0|(r_1^2 - \phi_M^2)}} \left[\chi - \frac{r_2^2 - \phi_M^2}{r_2^2} \Pi(\arcsin(\operatorname{sn}(\chi, k)), \check{\alpha}_3^2, k)\right], \end{aligned}$$
(6.4)

where $k^2 = \frac{r_1^2 - r_2^2}{r_1^2 - \phi_M^2}$, $\check{\alpha}_3^2 = \frac{k^2 \phi_M^2}{r_2^2}$. The level curves defined by $H_{-4}(\phi, y) = h_2$, contain two homoclinic orbits to two equilibrium points $(\pm \phi_2, 0)$ passing through two straight lines $\phi = \pm \sqrt{a_0}$, respectively and two heteroclinic orbits enclosing three equilibrium points $(\pm \phi_1, 0)$ and (0,0). For the right homoclinic orbit, we have $\sqrt{|h_2|}\xi = \int_{\phi}^{\phi_M} \frac{d\phi}{(\phi^2 - \phi_2^2)\sqrt{(\phi_M^2 - \phi^2)(\phi^2 + r_1^2)}}$ Thus, we obtain the parametric representations of two homoclinic solutions as follows:

$$\phi(\chi) = \pm \phi_M \operatorname{cn}(\chi, k), \quad \chi \in \left(-\operatorname{cn}^{-1}\left(\frac{\phi_2}{\phi_M}, k\right), \operatorname{cn}^{-1}\left(\frac{\phi_2}{\phi_M}, k\right)\right),$$

$$\xi(\chi) = \frac{1}{(\phi_M^2 - \phi_2^2)\sqrt{|h_2|(r_1^2 + \phi_M^2)}} \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \check{\alpha}_4^2, k),$$

$$\phi_M^2 = \chi^2 = \phi_M^2$$
(6.5)

where $k^2 = \frac{\phi_M^2}{r_1^2 + \phi_M^2}, \check{\alpha}_4^2 = \frac{\phi_M^2}{\phi_M^2 - \phi_2^2}.$

For the above heteroclinic orbit, we have $\sqrt{|h_2|}\xi = \int_0^\phi \frac{d\phi}{(\phi_2^2 - \phi^2)\sqrt{(\phi_M^2 - \phi^2)(\phi^2 + r_1^2)}}$ Hence, we obtain the parametric representations of two heteroclinic solutions as follows:

$$\begin{split} \phi(\chi) &= \pm \frac{r_1 k \operatorname{sn}(\chi, k)}{\operatorname{dn}(\chi, k)}, \quad \chi \in \left(-\operatorname{sn}^{-1} \left(\sqrt{\frac{\phi_2^2}{k^2 (r_1^2 + \phi_2^2)}}, k \right), \operatorname{sn}^{-1} \left(\sqrt{\frac{\phi_2^2}{k^2 (r_1^2 + \phi_2^2)}}, k \right) \right), \\ \xi(\chi) &= \frac{1}{(r_1^2 + \phi_2^2) \sqrt{|h_2|(r_1^2 + \phi_M^2)}} \left[\chi + \frac{r_1^2}{\phi_2^2} \Pi(\operatorname{arcsin}(\operatorname{sn}(\chi, k)), \check{\alpha}_5^2, k) \right], \end{split}$$

$$\begin{aligned} & \text{(6.6)} \end{split}$$

where $k^2 = \frac{\phi_M}{r_1^2 + \phi_M^2}$, $\check{\alpha}_5^2 = k^2 \left(1 + \frac{\prime_1}{\phi_2^2} \right)$.

6.2. The case of Figure.5 (b).

(i) For $\beta = -3$, we consider the case $0 < h_0 < h_1$. The level curves defined by $H_{-3}(\phi, y) = h < 0$ contain three families of periodic orbits for which one family enclose three equilibrium points (0,0) and $(\pm \phi_1, 0)$, other two families contact to two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \phi_2, 0)$, respectively, $\phi_2 = \sqrt{a_0}$. These families of orbits have the similar parametric representations as (5.6) and (5.7), where we use a_0 instead of r_2^2 .

The level curves defined by $H_{-3}(\phi, y) = h, h \in (0, h_0)$ contain a family of periodic orbits enclosing three equilibrium points (0,0) and $(\pm \phi_1, 0)$, and two open curve families contacting to two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \phi_2, 0)$, respectively. This family of orbits has the similar parametric representations as (6.1), where we use a_0 instead of r_1^2 .

The level curves defined by $H_{-3}(\phi, y) = h_0$ contain two homoclinic orbits of system (1.5) to the origin and two open curves contact to two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \phi_2, 0)$, respectively. Two homoclinic orbits have the similar parametric representations as (6.2), where we use a_0 instead of r_1^2 .

The level curves defined by $H_{-3}(\phi, y) = h, h \in (h_0, h_1)$ contain two families of closed orbits of system (1.5) enclosing the equilibrium points $(\pm \phi_1, 0)$ and two open curves contact to two straight lines $\phi = \pm \sqrt{a_0}$ at the points $(\pm \phi_2, 0)$. In this case, we have $\sqrt{h\xi} = \int_{r_2}^{\phi} \frac{d\phi}{\sqrt{(a_0 - \phi^2)(r_1^2 - \phi^2)(\phi^2 - r_2^2)}}$. Therefore, we have the parametric representations of two periodic families of system (1.5):

$$\phi(\xi) = \pm \left(\frac{r_2^2}{1 - \check{\alpha}_6^2 \mathrm{sn}^2(\check{\Omega}_2 \xi, k)}\right)^{\frac{1}{2}},\tag{6.7}$$

where $k^2 = \frac{(r_1^2 - r_2^2)a_0}{r_1^2(a_0 - r_2^2)}, \check{\alpha}_6^2 = \frac{r_1^2 - r_2^2}{r_1^2}, \check{\Omega}_2 = \sqrt{hr_1^2(a_0 - r_2^2)}.$

(ii) For $\beta = -4$, the level curves defined by $H_{-4}(\phi, y) = h_0 < 0$, contain two homoclinic orbits to the origin and two closed orbits contacting to two straight lines $\phi = \pm \sqrt{a_0}$, respectively. They have the similar parametric representations as (6.3) and (6.4), where we use a_0 instead of r_2^2 .

For the cases of Figure 5 (c) and Figure 5 (d), the orbits of system (1.5) have not given new forms of the parametric representations. So that, we do not make new discussion.

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