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CRITICAL POINT APPROACHES TO GRADIENT-TYPE SYSTEMS ON THE SIERPIŃSKI GASKET*

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Abstract We investigate the existence of multiple solutions for parametric quasi-linear systems of the gradient-type on the Sierpiński gasket. We give some new criteria to guarantee that the systems have at least three weak solutions by using a variational method and some critical points theorems due to Ricceri. We extend and improve some recent results. Finally, we give two examples to illustrate the main results.

Keywords Gradient-Type systems, Sierpiński gasket, nonlinear elliptic equation, variational methods, critical point theory.

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1. Introduction

In this paper, we are interested in Dirichlet gradient type system of the form:

$$\begin{cases} \Delta u_i(x) + a_i(x)u_i(x) = \lambda p(x)F_{u_i}(u_1(x),\dots,u_n(x)) \\ +\nu q(x)G_{u_i}(u_1(x),\dots,u_n(x)), \qquad\qquad x \in V/V_0 \\ u_i|_{V_0} = 0 \end{cases} \qquad (P^{F,G}_{\lambda,\nu})$$

for $i = 1,\dots,n$, where V stands for the Sierpiński gasket, V_0 is its intrinsic boundary, Δ denotes the weak Laplacian on V, λ and ν are positive real parameters, $F, G : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable in $(x_1,\dots,x_n)$ and $F(0,\dots,0) = G(0,\dots,0) = 0$, F_{u_i} and G_{u_i} denote the partial derivative of F and G with respect

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to u_i, respectively, the variable potentials $a_1, \dots, a_n, p, q : V \longrightarrow \mathbb{R}$ satisfy the following conditions:

(h_1) $a_i \in \mathrm{L}^1(V, \mu)$ and $a_i \leq 0$ $(i = 1, \dots, n)$ almost everywhere in V.

(h_2) $p, q \in \mathrm{C}(V)$ with $p \leq 0$ and $q \leq 0$ such that the restriction of p and q on every open subset of V is not identically zero.

Here μ denotes the restriction to V of the normalized $\frac{\log N}{\log 2}-$dimentional Hausdorff measure on V, so that $\mu(V) = 1$ (see [10] for more details).

The 'fractal' was originally due to Mandelbrot in 1975. A fractal often has the following properties: it has a simple recursive definition, it has a fine structure at arbitrary small scales, it is self-similar, and it has a Hausdorff dimension which is greater than its topological dimension. A simple example of a fractal is the Sierpiński gasket (triangle). It was introduced by Waclaw Sierpiński in [25], and plays an important role in the theory of curves. It is one of the basic examples of post critically finite fractals and the complement of it is a union of triangles [18].

Recently, there has been an increasing interest in studying nonlinear partial differential equations on fractals. One of the difficulties in studying PDEs on fractal domains is how to define differential operators, like the Laplacian, on the fractal domains. There is no concept of a generalized derivative of a function, and so we need to clarify the idea of differential operators such as the Laplacian on fractal domains. So, we cannot expect the solutions of PDEs on fractal domains to behave like the solutions of their Euclidean analogues. For example, Barlow and Kigami [2] proved that many fractals have Laplacian eigenfunctions vanishing identically on large open sets, whereas the eigenfunctions of the Laplace operator are analytic in $\mathbb{R}^n$. On the other hand, many researchers have used the variational method and critical point theory to investigate the nonlinear elliptic equations of fractals. We refer the readers to [6, 13, 14] for more details.

We refer to [27] for an elementary introduction to the Sierpiński Gasket, and [28] for important applications of fractals. Moreover, the study of the Laplacian on fractals originated in physics, literature, where the so-called spectral decimation method was developed in [1, 21]. For completeness, we recall that the Laplacian on the Sierpiński gasket was first constructed as the generator of a diffusion process (see [16, 19]). In [30], Teplyaev proved that the Laplacian with the Neumann boundary condition has pure point spectrum. Moreover, the set of eigenfunctions with compact support is complete. The same is true if the infinite Sierpiński gasket has no boundary, but is false for the Laplacian with the Dirichlet boundary condition.

The nonlinear problem $(P^{F,G}_{\lambda,\nu})$ is closely related to physical phenomena such as reaction-diffusion problems and elastic properties of fractal media and flow through fractal regions. There is an extensive theory for the study of nonlinear elliptic equations $(P^{F,G}_{\lambda,\nu})$ on classical domains, that is, on open sets of $\mathbb{R}^N$, using Sobolev spaces and Sobolev embedding theorems (see [3–5, 8, 12]). In [9], by extending a method introduced by Faraci and Kristály in the framework of Sobolev spaces to the case of function spaces on fractal domains, Breckner et. al established the existence of infinitely many weak solutions for the problem $\Delta u(x) + a(x)u(x) = g(x)f(u(x))$ in V/V_0 and $u|_{V_0} = 0$, where $a : V \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $g : V \to \mathbb{R}$ are continuous functions with appropriate properties. Moreover, in [26], Stancu-Dumitru studied the Dirichlet problem involving the weak Laplacian on the Sierpiński gasket $-\Delta u(x) = f(x)|u(x)|^{p-2}u(x) + (1 - g(x))|u(x)|^{q-2}u(x)$ in V/V_0 and $u|_{V_0} = 0$, where Δ is the Laplacian on V, $1 < p < 2 < q$ are real numbers, $f, g \in \mathrm{C}(V)$ satisfy

$f^+ = \max\{f, 0\} \neq 0$ and $0 \leq g(x) < 1$ for all $x \in V$, and proved the existence of at least two distinct nontrivial weak solutions using Ekeland's Variational Principle and standard tools in critical point theory combined with corresponding variational techniques. In [7], Bonanno et. al used variational methods to prove the existence of infinitely many solutions for a system of gradient type $(P^{F,G}_{\lambda,\nu})$, under an appropriate oscillating behavior either at zero or at infinity of the nonlinear data. Moreover, by adopting the same hypotheses on the potential and in presence of suitable small perturbations, the same conclusion is achieved.

In the present paper, we are interested to look for the existence of at least three weak solutions for the problem $(P^{F,G}_{\lambda,\nu})$ for appropriate values of the parameters λ and ν belonging to real intervals. Employing variational methods and two three critical points theorems due Ricceri [22, 23], we establish two existence results for the problem $(P^{F,G}_{\lambda,\nu})$. Two examples are presented to illustrate our main results.

2. Preliminaries

The proof of the main results are based on the following two three critical points theorem obtained by Ricceri in [22, 23]. Let X be a real Banach space, we use $\mathcal{W}_X$ to denote the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property: if $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

Remark 2.1. If X is uniformly convex and $g : [0, +\infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Theorem 2.1 ([22]). *Let X be a separable and reflexive real Banach space; let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1-functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, setting*

$$\rho = \max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\},$$

$$\sigma = \sup_{u\in\Phi^{-1}(0,+\infty)} \frac{J(u)}{\Phi(u)},$$

assume that $\rho < \sigma$. Then, for each compact interval $[c, d] \subset (\frac{1}{\sigma}, \frac{1}{\rho})$ (with the conventions $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$), there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every C^1-functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\gamma > 0$ such that, for each $\nu \in [0, \gamma]$,

$$\Phi'(u) = \lambda J'(u) + \nu\Psi'(u)$$

has at least three solutions in X whose norms are less than R.

Theorem 2.2 ([23]). *Let X be a reflexive real Banach space; $I \subseteq \mathbb{R}$ an interval; let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semi-continuous C^1-functional, bounded* *on each bounded subset of X, whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1-functional with compact derivative. Assume that*

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda J(u)) = +\infty$$

for all $\lambda \in I$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\in I}\inf_{u\in X}(\Phi(u) + \lambda(\rho - J(u))) < \inf_{u\in X}\sup_{\lambda\in I}(\Phi(u) + \lambda(\rho - J(u))).$$

Then, there exist a nonempty open set $A \subseteq I$ and a positive number R' with the following property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\nu \in [0, \delta]$, the equation $\Phi'(u) - \lambda J'(u) - \nu\Psi'(u) = 0$ has at least three solutions in X whose norms are less than R'.

Proposition 2.1 ([24]). *Let X be a nonempty set, and Φ and J be two real functions on X. Assume that there are $r > 0$ and $u_0, u_1 \in X$ such that*

$$\Phi(u_0) = J(u_0) = 0, \qquad \Phi(u_1) > r, \qquad \sup_{u\in\Phi^{-1}(-\infty,r]} J(u) < r\frac{J(u_1)}{\Phi(u_1)}.$$

Then for each ρ satisfying

$$\sup_{u\in\Phi^{-1}(-\infty,r]} J(u) < \rho < r\frac{J(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - J(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - J(u))).$$

We refer the reader to the paper [11,29] in which Theorems 2.1 and 2.2 were successfully employed to ensure the existence of at least three solutions for perturbed second-order Hamiltonian systems with impulsive effects. We also refer the readers to [10, 17] in which Theorem 2.1 was successfully employed to ensure the existence of three solutions for Dirichlet problem on the Sierpiński gasket and impulsive perturbed elastic beam fourth-order equations of Kirchhoff-type, respectively.

In this paper, we denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, ...\}$, by $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$ the set of positive naturals, and by $|\cdot|$ the Euclidian norm on the spaces $\mathbb{R}^n, n \in \mathbb{N}^*$. Now we give the two remarks below to more understanding about Sierpiński gasket.

Remark 2.2. The Sierpiński gasket is the connected subset of the plane obtained from an equilateral triangle by removing the open middle inscribed equilateral triangle of a quarter ($\frac{1}{4}$) of the area. Removing the corresponding open triangle from each of the three constituent triangles, and continuing this way. The gasket can also be obtained as the closure of the set of vertices arising in this construction.

Remark 2.3. Let $N \geq 2$ be a natural number and let $p_1, \ldots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$. Define, for every $i \in \{1, ..., N\}$, the map $S_i : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by $S_i(x) = \frac{1}{2}x + \frac{1}{2}p_i$. Obviously, every S_i is a similarity with ratio $\frac{1}{2}$. Let $\mathcal{S} := \{S_1, \ldots, S_N\}$ and set $F : \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ with $F(A) = \bigcup_{i=1}^{N} S_i(A)$. It is known (see [14, Theorem 9.1]) that there is a unique non-empty compact subset V of $\mathbb{R}^{N-1}$, called the attractor of the family $\mathcal{S}$, such that $F(V) = V$. The set V is called the Sierpiński gasket (SG for short) in $\mathbb{R}^{N-1}$. It can be constructed inductively as follows: Put $V_0 := \{p_1, ..., p_N\}, V_m := F(V_{m-1})$, for $m \geq 1$, and $V_* := \cup_{m\geq 0} V_m$. Since $p_i = S_i(p_i)$ for $i = 1, ..., N$, we have $V_0 \subseteq V_1$ and $F(V_*) = V_*$. Taking into account that the maps $S_i, i = 1, ..., N$, are homeomorphisms, we conclude that V_* is a fixed point of F. On the other hand, $\overline{V_*}$ is non-empty, compact and $V = \overline{V_*}$. The set V_0 is called the intrinsic boundary of the SG. The Hausdorff dimension d of V satisfies the equality $\sum_{i=1}^{N} (\frac{1}{2})^d = 1$ (see [14, Theorem 9.3]). Hence $d = \frac{\ln N}{\ln 2}$, and $0 < \mathcal{H}^d(V) < \infty$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$. Let μ be the normalized restriction of $\mathcal{H}^d$ to the subsets of V, and so $\mu(V) = 1$. Moreover $\mu(B) > 0$ for every nonempty open subset B of V. In other words, the support of μ coincides with V.

Now, denote by $C(V)$ the space of real-valued continuous functions on V and

$$C_0(V) := \{u \in C(V);\ u|_{V_0} = 0\}.$$

The space $C(V)$ and $C_0(V)$ endowed with the usual supremum norm $\|\cdot\|_\infty$. For a function $u : V \longrightarrow \mathbb{R}$ and for $m \in N$, let

$$W_m = (\frac{N+2}{N})^m \sum_{x,y\in V_m, |x-y|=2^{-m}} [u(x) - u(y)]^2. \tag{2.1}$$

We have $W_m(u) \leq W_{m+1}(u)$ for every natural m. So we can put

$$W(u) = \lim_{m\to\infty} W_m(u). \tag{2.2}$$

Define

$$H_0^1(V) := \{u \in C_0(V);\ W(u) < \infty\}.$$

It turns out $H_0^1(V)$ is a dense linear subset of $L^2(V, \mu)$ equipped with the $\|\cdot\|_2$ norm. We endow $H_0^1(V)$ with the norm

$$\|u\| = \sqrt{W(u)}.$$

In fact, there is an inner product defining this norm: for $u, v \in H_0^1(V)$ and $m \in N$, let

$$\mathcal{W}_m = \left(\frac{N+2}{N}\right)^m \sum_{x,y\in V_m, |x-y|=2^{-m}} (u(x) - u(y))(v(x) - v(y)).$$

Put

$$\mathcal{W}(u, v) = \lim_{m\to\infty} \mathcal{W}_m(u, v).$$

Then, $W(u, v) \in \mathbb{R}$ and $H_0^1(V)$, equipped with the inner product W (which obviously induces the norm $\|\cdot\|$) becomes real Hilbert space. Moreover,

$$\|u\|_\infty \leq (2N + 3)\|u\|, \quad \text{for all}\, u \in H_0^1(V), \tag{2.3}$$

and the embedding

$$(H_0^1(V), \|\cdot\|) \hookrightarrow (C_0(V), \|\cdot\|_\infty), \tag{2.4}$$

is compact. We refer to [15] for further details.

We now define Laplacian on the Sierpiński gasket V. Let $H^{-1}(V)$ be the closure of $\mathrm{L}^2(\mathrm{V})$ with respect to the pre-norm

$$\|w\|_{-1} = \sup_{w\in H_0^1(V), \|g\|=1} |\langle w, g\rangle|$$

where

$$\langle w, g\rangle = \int_V wg\mathrm{d}\mu,$$

for $w \in \mathrm{L}^2(V)$ and $g \in H_0^1(V)$. Then $H^{-1}(V)$ is a Hilbert space. Let $W(u, v)$ be the inner product of $u, v \in H_0^1(V)$. Then the relation

$$-W(u, v) = \langle \Delta u, v\rangle, \quad \text{for all}\, v \in H_0^1(V),$$

uniquely defines a function $\Delta u \in H^{-1}(V)$ for all $u \in H_0^1(V)$; we term Δ the (weak) Laplacian on V (see [20]).

Remark 2.4. As pointed out by Falconer and Hu [14], we just observe that if $a \in L^1(V)$ and $a \leq 0$ in V, then from (2.3), the norm

$$\|u\|_* := \left(W(u, u) - \int_V a(x)u^2(x)\mathrm{d}\mu\right)^{\frac{1}{2}},$$

is equivalent to $\sqrt{W(u)}$ in $H_0^1(V)$.

Fix $\lambda > 0$. We say that a function $(u_1, \dots, u_2) \in H_0^1(V) \times H_0^1(V)$ is called a weak solution of $(P_{\lambda,\nu}^{F,G})$ if

$$\sum_{i=1}^{n}\left[(W(u_i, v_i) - \int_V a_i(x)u_i(x)v_i(x)\mathrm{d}\mu) + \lambda\int_V g(x)F_{u_i}(u_1(x), u_2(x))v_i(x)\mathrm{d}\mu\right] = 0$$

for every $(v_1, v_2) \in H_0^1(V) \times \ldots \times H_0^1(V)$.

Remark 2.5. If $a_1, a_2 \in \mathrm{C}(V)$, arguing as in Lemma 2.16 of [14], it follows that every weak solution of the problem $(P_{\lambda,\nu}^{F,G})$ is also a strong solution.

Here and in the sequel, E will denote the product space $E = H_0^1(V)\times \ldots, \times H_0^1(V)$ endowed with the norm

$$\|u\|_{\mathrm{E}} = \|(u_1, u_2 \ldots, u_n)\|_{\mathrm{E}} := \sum_{i=1}^{n}\left(W(u_i) - \int_V a_i(x)u_i^2(x)\mathrm{d}\mu\right)^{\frac{1}{2}}.$$

We say that a function $u = (u_1, \dots, u_n) \in H_0^1(V) \times \ldots, \times H_0^1(V)$ is called a weak solution of $(P_{\lambda,\nu}^{F,G})$ if

$$\begin{aligned}
&\sum_{i=1}^{n}\left[W(u_i, v_i) - \int_V a_i(x)u_i(x)v_i(x)\mathrm{d}\mu\right] \\
&+ \lambda\int_V \sum_{i=1}^{n} p(x)F_{u_i}(u_1(x), \dots, u_n(x))v_i(x)\mathrm{d}\mu \\
&+ \nu\int_V \sum_{i=1}^{n} q(x)G_{u_i}(u_1(x), \dots, u_n(x))v_i(x)\mathrm{d}\mu = 0
\end{aligned}$$

for every $v = (v_1, \ldots, v_n) \in H_0^1(V) \times \ldots, \times H_0^1(V)$.

Now for every $u = (u_1, \ldots, u_n) \in \mathrm{E}$, we define

$$\Phi(u) = \frac{1}{2}\sum_{i=1}^{n}\Big(\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x)u_i^2(x)\mathrm{d}\mu\Big) \tag{2.5}$$

$$J(u) = \int_V p(x)F(u_1(x), \ldots, u_n(x))\mathrm{d}\mu \tag{2.6}$$

and

$$\Psi(u) = \int_V q(x)G(u_1(x), \ldots, u_n(x))\mathrm{d}\mu. \tag{2.7}$$

Standard arguments show that $I =: \Phi - \mu\Psi - \lambda J$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u = (u_1, \ldots, u_n) \in \mathrm{E}$ given by

$$\begin{aligned} I'(u)(v) = &\sum_{i=1}^{n}\Big[W(u_i, v_i) - \int_V a_i(x)u_i(x)v_i(x)\mathrm{d}\mu\Big] \\ &+ \lambda\int_V p(x)\sum_{i=1}^{n}F_{u_i}(u_1(x), \ldots, u_n(x))v_i(x)\mathrm{d}\mu \\ &+ \nu\int_V\sum_{i=1}^{n}q(x)G_{u_i}(u_1(x), \ldots, u_n(x))v_i(x)\mathrm{d}\mu = 0 \end{aligned}$$

for all $v = (v_1, \ldots, v_n) \in \mathrm{E}$. We observe that a vector $u \in \mathrm{E}$ is a solution of problem $(P^{F,G}_{\lambda,\nu})$ if and only if u is a critical point of the function I.

Proposition 2.2. *Let $\mathcal{J} := \Phi' : \mathrm{E} \longrightarrow \mathrm{E}^*$ be the operator defined by*

$$\mathcal{J}(u)(v) = \sum_{i=1}^{n}\Big[W(u_i, v_i) - \int_V a_i(x)u_i(x)v_i(x)\mathrm{d}\mu\Big]$$

for every $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathrm{E}$. Then J admits a continuous inverse on X^*.

Proof. We should show that $\mathcal{J}$ is strictly monotone and coercive operator. we have

$$\begin{aligned} &\langle\mathcal{J}(u_1, \ldots, u_n)(u_1 - v_1, \ldots, u_n - v_n) - \mathcal{J}(v_1, \ldots, v_n)(u_1 - v_1, \ldots, u_n - v_n)\rangle \\ &= \sum_{i=1}^{n}(W(u_i, u_i - v_i) - \int_V a_iu_i(u_i - v_i)\mathrm{d}\mu) \\ &- \sum_{i=1}^{n}(W(v_i, u_i - v_i) - \int_V a_iv_i(u_i - v_i)\mathrm{d}\mu) \\ &= \sum_{i=1}^{n} - \int_V a_i(x)u_i(u_i - v_i)\mathrm{d}\mu + \sum_{i=1}^{n}\int_V a_i(x)v_i(u_i - v_i)\mathrm{d}\mu \\ &= -\sum_{i=1}^{n}\int_V a_i(u_i - v_i)(u_i - v_i)\mathrm{d}\mu \\ &= -\sum_{i=1}^{n}\int_V a_i(x)(u_i - v_i)^2\mathrm{d}\mu. \end{aligned}$$

Since $a_i \in \mathrm{L}^1(V,\mu)$ and $a_i \le 0$, $i = 1,\dots,n$ then

$$\begin{aligned}
&\langle \mathcal{J}(u_1,\dots,u_n)(u_1-v_1,\dots,u_n-v_n) - \mathcal{J}(v_1,\dots,v_2)(u_1-v_1,\dots,u_n-v_n)\rangle \\
&\ge \tau \left[\int_V (u_1-v_1)^2 \mathrm{d}\mu + \dots + \int_V (u_2-v_2)^2 \mathrm{d}\mu\right] \\
&= \tau \|u-v\|^2,
\end{aligned}$$

where $\tau = \sum_{i=1}^n \int_V a_i \, d\mu$, and so $\mathcal{J}$ is a strictly monotone and coercive operator. Then $\mathcal{J}$ admits a continuous inverse on E^*. $\square$

3. Main results

In this section, we formulate our main results. Let us denote by $\mathcal{F}$ the class of all functions $F : \mathbb{R}^n \to \mathbb{R}$ that are continuously differentiable in ξ satisfy the standard summability condition

$$\sup_{|\xi|\le \varrho_1} \left\{ \max\{|F(\xi)|,\ |G(\xi)|,\ |F_{\xi_i}(\xi)|,\ |G_{\xi_i}(\xi)|,\ i=1,\dots,n\} \right\} \in \mathrm{L}^1(V,\mu) \tag{3.1}$$

for any $\varrho_1 > 0$ with $\xi = (\xi_1,\dots,\xi_n)$.

Let

$$\lambda_1 = \inf\left\{ \frac{\sum_{i=1}^n \left(\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x)u_i^2(x)\mathrm{d}\mu\right)}{2\int_\Omega p(x)F(u_1(x),\dots,u_n(x))\mathrm{d}x} : \right. \\ \left. u \in \mathrm{E},\ \int_\Omega p(x)F(u_1(x),\dots,u_n(x))\mathrm{d}\mu > 0 \right\}$$

and $\lambda_2 = \frac{1}{\max\{0,\lambda_0,\lambda_\infty\}}$, where

$$\lambda_0 = \limsup_{|u|\to 0} \frac{2\int_\Omega p(x)F(u_1(x),\dots,u_n(x))\mathrm{d}\mu}{\sum_{i=1}^n \left(\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x)u_i^2(x)\mathrm{d}\mu\right)},$$

and

$$\lambda_\infty = \limsup_{\|u\|\to +\infty} \frac{2\int_\Omega p(x)F(u_1(x),\dots,u_n(x))\mathrm{d}\mu}{\sum_{i=1}^n \left(\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x)u_i^2(x)\mathrm{d}\mu\right)},$$

where $u = (u_1,\dots,u_n)$ and $|u| = \left(u_1^2 + \dots + u_n\right)^{\frac{1}{2}}$,

Theorem 3.1. *Let $F \in \mathcal{F}$ and $p : V \to \mathbb{R}$ satisfy in the assumption (h_2). Assume that*

$(\mathcal{A}_1)$ *there exists a constant $\varepsilon > 0$ such that*

$$\sup_{x\in V} p(x). \max\left\{ \limsup_{(u_1,\dots,u_n)\to(0,\dots,0)} \frac{F(u_1(x),\dots,u_n(x))}{|u|^p}, \right. \\ \left. \limsup_{|u|\to+\infty} \frac{\max_{x\in\overline{\Omega}} p(x)F(u_1(x),\dots,u_n(x))}{|u|^p} \right\} < \varepsilon$$

where $|u| = \sqrt{\sum_{i=1}^n u_i^2}$;

($\mathcal{A}_2$) *there exists a function* $w = (w_1, \dots, w_n) \in \mathrm{E}$ *such that*

$$K_w := \sum_{i=1}^{n} \Big(\|w_i\|^2_{H_0^1(V)} - \int_V a_i(x) w_i^2(x) \mathrm{d}\mu \Big) \neq 0$$

and

$$\varepsilon < \frac{2 \int_\Omega p(x) F(w_1(x), \dots, w_n(x)) \mathrm{d}\mu}{(2N+3)^2 K_w}.$$

Then, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$*, there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and, for every* $G \in \mathcal{F}$ *and every* $q : V \to \mathbb{R}$ *satisfy in the assumption* (h_2) *there exists* $\gamma > 0$ *such that for each* $\nu \in [0, \gamma]$*, the problem* ($P^{F,G}_{\lambda,\nu}$) *has at least three weak solutions whose norms in* E *are less than* R.

Proof. Take $X = \mathrm{E}$. It is clear that E is a separable and uniformly convex Banach space. Let the functionals Φ, J and Ψ be as given in (2.5), (2.6) and (2.7), respectively. The functional Φ is C^1, and by Proposition 2.2, its derivative admits a continuous inverse on X^*. Moreover, clearly, Φ is coercive and sequentially weakly lower semicontinuous functional. Furthermore, let A be a bounded subset of X. That is, there exist constant $c > 0$, such that $\|u\|_{\mathrm{E}} \leq c$ for each $u = (u_1, \dots, u_n) \in A$. Then, by (2.5) there exist constants $c_1, c_2, \dots, c_n$ such that $c = \sum_{i=1}^n c_i$ and $\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x) u_i^2(x) \mathrm{d}\mu \leq c_i^2$ for $i = 1, \dots, n$, and we have

$$|\Phi(u)| \leq \frac{1}{2} \sum_{i=1}^{n} \Big(\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x) u_i^2(x) \mathrm{d}\mu \Big) \leq \frac{1}{2} \sum_{i=1}^{n} c_i^2.$$

Hence Φ is bounded on each bounded subset of X. Furthermore, by Remark 2.1, $\Phi \in \mathcal{W}_X$. The functionals J and Ψ are two C^1-functionals with compact derivatives. Moreover, Φ has a strict local minimum 0 with $\Phi(0) = J(0) = 0$. In view of ($\mathcal{A}_1$), there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(u_1, \dots, u_n) \leq \varepsilon |u|^2 \tag{3.2}$$

for every u with $|u| \in [0, \tau_1) \cup (\tau_2, +\infty)$, where $|u| = \sqrt{\sum_{i=1}^n u_i^2}$. Since $F(u_1, \dots, u_n)$ is continuous on $\mathbb{R}^n$, it is bounded on $|u| = \sqrt{\sum_{i=1}^n u_i^2} \in [\tau_1, \tau_2]$. Thus we can choose $\eta > 0$ and $\upsilon > 2$ such that

$$F(u_1, \dots, u_n) \leq \varepsilon |u|^2 + \eta |u|^\upsilon$$

for all $(x, u_1, \dots, u_n) \in V \times \mathbb{R}^n$. So, by (2.3) and Remark 2.4, we have

$$\begin{aligned} J(u) \leq & \varepsilon \int_V \sum_{i=1}^n u_i^2 \mathrm{d}\mu + \eta \int_V \Big(\sum_{i=1}^n u_i^2 \Big)^{\frac{\nu}{2}} \mathrm{d}\mu \\ \leq & \varepsilon (2N+3)^2 \Big(\int_V p(x) \mathrm{d}\mu \Big) \|u\|^2_{\mathrm{E}} + \eta (2N+3)^{\frac{\upsilon}{2}} \Big(\int_V p(x) \mathrm{d}\mu \Big) \|u\|^\upsilon_{\mathrm{E}} \end{aligned} \tag{3.3}$$

for all $u = (u_1, \dots, u_n) \in X$. Hence, from (3.3) we have

$$\limsup_{|u| \to 0} \frac{J(u)}{\Phi(u)} \leq \frac{\varepsilon (2N+3)^2 \Big(\int_V p(x) \mathrm{d}\mu \Big)}{2}. \tag{3.4}$$

Moreover, by using (3.2), for each $u=(u_1,\dots,u_n)\in \mathrm{E}\setminus\{(0,0,\dots,0)\}$, we obtain

$$\begin{aligned}\frac{J(u)}{\Phi(u)} &= \frac{\int_{|u|\le\tau_2} p(x)F(u_1,\dots,u_n)\mathrm{d}\mu}{\Phi(u)} + \frac{\int_{|u|>\tau_2} p(x)F(u_1,\dots,u_n)\mathrm{d}\mu}{\Phi(u)}\\ &\le \frac{\sup_{x\in V} p(x).\sup_{|u|\in[0,\tau_2]} F(u_1,\dots,u_n)}{\Phi(u)} + \frac{\varepsilon(2N+3)^2\Big(\int_V p(x)\mathrm{d}\mu\Big)\|u\|_{\mathrm{E}}^2}{\Phi(u)}\\ &\le \frac{2\sup_{x\in\Omega,|u|\in[0,\tau_2]} F(x,u)}{\|u\|^2} + \frac{\varepsilon(2N+3)^2\Big(\int_V p(x)\mathrm{d}\mu\Big)}{2}.\end{aligned}$$

So, we get

$$\limsup_{\|u\|\to+\infty}\frac{J(u)}{\Phi(u)} \le \frac{\varepsilon(2N+3)^2\Big(\int_V p(x)\mathrm{d}\mu\Big)}{2}. \tag{3.5}$$

In view of (3.4) and (3.5), we have

$$\rho=\max\left\{0,\ \limsup_{\|u\|\to+\infty}\frac{J(u)}{\Phi(u)},\ \limsup_{(u_1,\dots,u_n)\to(0,\dots,0)}\frac{J(u)}{\Phi(u)}\right\} \le \frac{\varepsilon(2N+3)^2\Big(\int_V p(x)\mathrm{d}\mu\Big)}{2}, \tag{3.6}$$

where $u=(u_1,\dots,u_n)$. Assumption $(\mathcal{A}_2)$ in conjunction with (3.6) yields

$$\begin{aligned}\sigma &= \sup_{u\in\Phi^{-1}(0,+\infty)}\frac{J(u)}{\Phi(u)} = \sup_{X\setminus\{(0,\dots,0)\}}\frac{J(u)}{\Phi(u)} \ge \frac{\int_V p(x)F(w_1(x),\dots,w_n(x))\mathrm{d}\mu}{\Phi(w(x))}\\ &= \frac{\int_\Omega p(x)F(w_1(x),\dots,w_n(x))\mathrm{d}x}{K_w} > \frac{\varepsilon(2N+3)^2\Big(\int_V p(x)\mathrm{d}\mu\Big)}{2} \ge \rho.\end{aligned}$$

Thus, all the hypotheses of Theorem 2.1 are satisfied. Clearly, $\lambda_1=\frac{1}{\sigma}$ and $\lambda_2=\frac{1}{\rho}$. Then, using Theorem 2.1, for each compact interval $[c,d]\subset(\lambda_1,\lambda_2)$, there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and, for every $G\in\mathcal{F}$ and every $q:V\to\mathbb{R}$ satisfy in the assumption (h_2) there exists $\gamma>0$ such that for each $\nu\in[0,\gamma]$, the system $(P^{F,G}_{\lambda,\nu})$ has at least three solutions whose norms in X are less than R. $\square$

Theorem 3.2. *Let $F\in\mathcal{F}$, $p:V\to\mathbb{R}$ satisfy in the assumption (h_2),*

$$\sup_{x\in V} p(x).\max\left\{\limsup_{(u_1,\dots,u_n)\to(0,\dots,0)}\frac{F(u_1(x),\dots,u_n(x))}{|u|^p},\ \limsup_{|u|\to+\infty}\frac{F(u_1(x),\dots,u_n(x))}{|u|^p}\right\}\le 0 \tag{3.7}$$

where $|u|=\sqrt{\sum_{i=1}^n u_i^2}$, and

$$\sup_{u\in\mathrm{E}}\frac{2\int_\Omega p(x)F(u_1(x),\dots,u_n(x))\mathrm{d}\mu}{\sum_{i=1}^n\Big(\|u_i\|^2_{H_0^1(V)}-\int_V a_i(x)u_i^2(x)\mathrm{d}\mu\Big)}>0 \tag{3.8}$$

where $u=(u_1,\dots,u_n)$. Then for each compact interval $[c,d]\subset(\lambda_1,+\infty)$ there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and, for every $G\in\mathcal{F}$

and every $q : V \to \mathbb{R}$ satisfy in the assumption (h_2) there exists $\gamma > 0$ such that for each $\nu \in [0, \gamma]$, the problem $(P^{F,G}_{\lambda,\nu})$ has at least three weak solutions whose norms in E *are less than R.*

Proof. In view of (3.7), there exist an arbitrary $\varepsilon > 0$ and τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(u_1, \dots, u_n) \leq \varepsilon |u|^2$$

for every u with $|u| \in [0, \tau_1) \cup (\tau_2, +\infty)$. Since $F(u_1, \dots, u_n)$ is continuous on $\mathbb{R}^n$, it is bounded on $|u| \in [\tau_1, \tau_2]$. Thus we can choose $\eta > 0$ and $\upsilon > p$ in a manner that

$$F(u_1, \dots, u_n) \leq \varepsilon |u|^2 + \eta |u|^\upsilon$$

for all $(u_1, \dots, u_n) \in \mathbb{R}^n$. So, by the same process in proof of Theorem 3.1 we have Relations (3.4) and (3.5). Since ε is arbitrary, (3.4) and (3.5) gives

$$\max \left\{ 0, \limsup_{\|u\| \to +\infty} \frac{J(u)}{\Phi(u)}, \limsup_{(u_1, \dots, u_n) \to (0, \dots, 0)} \frac{J(u)}{\Phi(u)} \right\} \leq 0$$

where $u = (u_1, \dots, u_n)$. Then, with the notation of Theorem 2.1, we have $\rho = 0$. By (3.8), we also have $\sigma > 0$. In this case, clearly $\lambda_1 = \frac{1}{\sigma}$ and $\lambda_2 = +\infty$. Thus, by using Theorem 2.1 result is achieved. □

Now we formulate the following applications of Theorems 2.2 as our second main result.

Theorem 3.3. *Assume that*

$(\mathcal{B}_1)$ *there exist two positive constants κ, ξ and $\alpha \in [0, 2)$ such that*

$$|F(u_1, \dots, u_n)| \leq \kappa |u|^\alpha + \xi \quad \text{for all}\, u = (u_1, \dots, u_n) \in \mathbb{R}^n$$

where $|u| = \sqrt{\sum_{i=1}^n u_i^2}$;

$(\mathcal{B}_2)$ *there exist a positive constant r and $w = (w_1, \dots, w_n) \in$ E such that $K_w > r$ where K_w is as given in Assumption $(\mathcal{A}_2)$ in Theorem 3.1, and*

$$\max_{|u| \leq \frac{\sqrt{2r}}{2N+3}} F(u_1, \dots, u_n) < \frac{r \int_V p(x) F(w_1(x), \dots, w_n(x)) \mathrm{d}\mu}{K_w \int_V p(x) \mathrm{d}\mu}.$$

Then, there exist a nonempty open set $A \subset [0, +\infty)$ and a positive number R' with the following property: for every $\lambda \in A$ and, for every $G \in \mathcal{F}$ and every $q : V \to \mathbb{R}$ satisfy in the assumption (h_2) there exists $\delta > 0$ such that for each $\nu \in [0, \delta]$, the problem $(P^{F,G}_{\lambda,\nu})$ has at least three weak solutions whose norms in E *are less than R'.*

Proof. Take $X =$ E. Let the functionals Φ and J be as given in (2.5) and (2.6), respectively. For any $\lambda \geq 0$ and $u = (u_1, \dots, u_n) \in$ E with $|u| = \sqrt{\sum_{i=1}^n u_i^2}$, by

Remark 2.4, (2.3) and $(\mathcal{B}_1)$ we have

$$\begin{aligned}\Phi(u) - \lambda J(u) &= \frac{1}{2}\sum_{i=1}^{n}\left(\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x)u_i^2(x)\mathrm{d}\mu\right) \\ &\quad - \lambda\int_V p(x)F(u_1(x),\ldots,u_n(x))\mathrm{d}\mu \\ &\geq \frac{1}{2}\sum_{i=1}^{n}\left(\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x)u_i^2(x)\mathrm{d}\mu\right) \\ &\quad - \lambda\int_V \left(\kappa p(x)|u(x)|^\alpha + \xi p(x)\right)\mathrm{d}\mu \\ &\geq \frac{1}{2}\sum_{i=1}^{n}\left(\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x)u_i^2(x)\mathrm{d}\mu\right) \\ &\quad - \lambda\left[(2N+3)^\alpha\kappa\|u\|^\alpha_{\mathrm{E}} - \xi\right]\int_V p(x)\mathrm{d}\mu.\end{aligned}$$

Since $\alpha < 2$ and $\|u\|_{\mathrm{E}} = \sum_{i=1}^{n}\left(\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x)u_i^2(x)\mathrm{d}\mu\right)^{\frac{1}{2}}$, one has

$$\lim_{\|u\|_{\mathrm{E}}\to+\infty}(\Phi(u) - \lambda J(u)) = +\infty$$

for all $\lambda \geq 0$. If $\Phi(u) \leq r$, we have $\|u\| \leq \sqrt{2r}$, that is,

$$\Phi^{-1}(-\infty, r] \subseteq \left\{u \in X : \max_{x\in V}|u(x)| \leq \frac{\sqrt{2r}}{2N+3}\right\}.$$

Therefore,

$$\begin{aligned}\sup_{u\in\Phi^{-1}(-\infty,r]} J(u) &\leq \max_{|u|\leq\frac{\sqrt{2r}}{2N+3}} J(u) \\ &= \max_{|u|\leq\frac{\sqrt{2r}}{2N+3}}\int_V p(x)F(u_1(x),\ldots,u_n(x))\mathrm{d}\mu \\ &\leq \max_{|u|\leq\frac{\sqrt{2r}}{2N+3}} F(u_1,\ldots,u_n).\int_V p(x)\mathrm{d}\mu.\end{aligned} \tag{3.9}$$

It is clear that $\Phi(0,\ldots,0) = J(0\ldots,0) = 0$ and owing to $(\mathcal{B}_2)$ and (3.9), $\Phi(w) = \Phi(w_1,\ldots,w_n) > r$ and

$$\sup_{u\in\Phi^{-1}(-\infty,r]} J(u) < r\frac{J(w)}{\Phi(w)}.$$

Thus we can fix ρ such that

$$\sup_{u\in\Phi^{-1}(-\infty,r]} J(u) < \rho < r\frac{J(w)}{\Phi(w)}.$$

Now from Proposition 2.1, we obtain

$$\sup_{\lambda\geq0}\inf_{u\in\mathrm{E}}(\Phi(u) + \lambda(\rho - J(u))) < \inf_{u\in\mathrm{E}}\sup_{\lambda\geq0}(\Phi(u) + \lambda(\rho - J(u))).$$

Therefore, by Theorem 2.2, and for each compact interval $[a,b] \subseteq (\lambda_1, \lambda_2)$, there exists $R' > 0$ with the following property: for every $\lambda \in [a,b]$, for every $G \in \mathcal{F}$ and every $q : V \to \mathbb{R}$ satisfy in the assumption (h_2) there exists $\delta > 0$ such that, for each $\nu \in [0,\delta]$, $\Phi'(u) - \lambda J'(u) - \mu\Psi'(u) = 0$ has at least three solutions in E. Hence, the problem $(P^{F,G}_{\lambda,\nu})$ has at least three weak solutions whose norms are less than R'. $\square$

Here $B(x_0, s)$ denotes the ball with center at x_0 and radius of s. Put

$$\vartheta(x) = \sup\{\vartheta > 0 : \ B(x,\vartheta) \subseteq V\}$$

for all $x \in V$, one can prove that there exists $x_0 \in V$ such that $B(x_0, D) \subseteq V$, where $D := \sup_{x\in V} \vartheta(x)$. Put

$$L := 2D^{N-2}(2^N - 1)\pi^{N/2} \sum_{i=1}^{n} \|a_i\|_{\mathrm{L}^1(V)}. \tag{3.10}$$

The next two theorems provide sufficient conditions for applying Theorems 3.1 and 3.3 which does not require to know a test function $w = (w_1, \dots, w_n)$ satisfying $(\mathcal{A}_2)$ and $(\mathcal{B}_2)$, respectively.

Theorem 3.4. *Assume that the assumption $(\mathcal{A}_1)$ in Theorem 3.1 holds and there exists a positive constant b such that*

$(\mathcal{A}_3)$ *$p(x) \geq 0$ for each $x \in B(x_0, D) \setminus B(x_0, \frac{D}{2})$ and $F(t_1, \dots, t_n) \geq 0$ for each $(t_1, \dots, t_n) \in \mathbb{R}^n$ with $|t| = \sqrt{\sum_{i=1}^n t_i^2} \in [0, b]$;*

$(\mathcal{A}_4)$ *there is $j \in \{1, \dots, n\}$ such that $a_j \neq 0$ and*

$$\varepsilon < \frac{\Gamma(1 + N/2) F(b, \dots, b) \int_{B(x_0, \frac{D}{2})} p(x) \mathrm{d}\mu}{b^2 L (2N+3)^2}$$

where Γ is the Gamma function and L is given in (3.10).

Then, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and, for every $G \in \mathcal{F}$ and every $q : V \to \mathbb{R}$ satisfy in the assumption (h_2) there exists $\gamma > 0$ such that, for each $\nu \in [0,\gamma]$, the problem $(P^{F,G}_{\lambda,\nu})$ has at least three weak solutions whose norms in E *are less than R.*

Proof. We claim that all the assumptions of Theorem 3.1 are fulfilled by choosing $w = (w_1(x), \dots, w_n(x))$ as follows

$$w_i(x) = \begin{cases} 0, & x \in V \backslash B(x_0, D) \\ \frac{2b}{D}(D - |x_i - x_0|), & x \in B(x_0, D) \backslash B(x_0, \frac{D}{2}) \\ b, & x \in B(x_0, \frac{D}{2}) \end{cases} \tag{3.11}$$

for $i = 1, \dots, n$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. It is easy to see that

$w \in \mathrm{E}$, and one has

$$\begin{aligned}\Phi(w) &= \frac{1}{2}\sum_{i=1}^{n}\int_{B(x_0,D)\setminus B(x_0,D/2)} a_i(x)\frac{(2b)^2}{D^2}\mathrm{d}\mu \\ &= \frac{\mathrm{meas}(B(x_0,D)) - \mathrm{meas}(B(x_0,D/2))}{2}\sum_{i=1}^{n}\|a_i\|_{\mathrm{L}^1(V)}\frac{(2b)^2}{D^2} \\ &= \frac{2b^2D^{N-2}(2^N-1)\pi^{N/2}}{\Gamma(1+N/2)}\sum_{i=1}^{n}\|a_i\|_{\mathrm{L}^1(V)} = \frac{Lb^2}{\Gamma(1+N/2)}.\end{aligned} \tag{3.12}$$

From Assumptions $(\mathcal{A}_3)$ and $(\mathcal{A}_4)$ we observe that Assumption $(\mathcal{A}_2)$ in Theorem 3.1 is satisfied. Hence, Theorem 3.1 follows the result. $\square$

Theorem 3.5. *Assume that Assumption $(\mathcal{B}_1)$ in Theorem 3.3 and Assumption $(\mathcal{A}_3)$ in Theorem 3.4 hold and there exist three positive constants b, c and α with $\alpha \in [0,2)$, and $\sqrt{2L}b < (2N+3)c\sqrt{\Gamma(1+N/2)}$ where Γ is the Gamma function and L is given in (3.10) such that*

$(\mathcal{B}_3)$ $\max_{|u|\le c} F(u_1,\dots,u_n) < \frac{F(b,\dots,b)}{2\int_V p(x)\mathrm{d}\mu}\int_{B(x_0,\frac{D}{2})} p(x)\mathrm{d}\mu.$

Then, there exist a nonempty open set $A \subset [0,+\infty)$ and a positive number R' with the following property: for every $\lambda \in A$ and, for every $G \in \mathcal{F}$ and every $q : V \to \mathbb{R}$ satisfy in the assumption (h_2) there exists $\delta > 0$ such that, for each $\nu \in [0,\delta]$, the problem $(P^{F,G}_{\lambda,\nu})$ has at least three weak solutions whose norms in E *are less than R'.*

Proof. We claim that all the hypotheses of Theorem 3.1 are satisfied by choosing $w = (w_1,\dots,w_n)$ as given in (3.11) and $r = \frac{Lb^2}{2\Gamma(1+N/2)}$. We observe that

$$K_w = \frac{2b^2D^{N-2}(2^N-1)\pi^{N/2}}{\Gamma(1+N/2)}\sum_{i=1}^{n}\|a_i\|_{\mathrm{L}^1(V)} = \frac{Lb^2}{\Gamma(1+N/2)} > \frac{Lb^2}{2\Gamma(1+N/2)} = r,$$

where K_w is as given in Assumption $(\mathcal{A}_1)$. Owing to $(\mathcal{B}_3)$ and $F(0,\dots,0) = 0$, one has $\int_{B(x_0,\frac{D}{2})} F(w_1(x),\dots,w_n(x))\mathrm{d}\mu > 0$. So by $(\mathcal{A}_3)$, $(\mathcal{B}_3)$ and (3.12), we have

$$\begin{aligned}& r\frac{\int_V p(x)F(w_1(x),\dots,w_n(x))\mathrm{d}\mu}{K_w\int_V p(x)\mathrm{d}\mu} \\ &= \frac{Lb^2}{2\Gamma(1+N/2)}\frac{\Gamma(1+N/2)\int_V p(x)F(w_1(x),\dots,w_n(x))\mathrm{d}\mu}{Lb^2\int_V p(x)\mathrm{d}\mu} \\ &> \frac{F(b,\dots,b)}{2\int_V p(x)\mathrm{d}\mu}\int_{B(x_0,\frac{D}{2})} p(x)\mathrm{d}\mu \\ &> \max_{|u|\le c} F(u_1,\dots,u_n) \\ &> \max_{|u|\le \frac{\sqrt{2r}}{2N+3}} F(u_1,\dots,u_n).\end{aligned}$$

Thus, the assumption $(\mathcal{B}_2)$ in Theorem 3.3 holds. Therefore, by Theorem 3.3, for each compact interval $[a,b] \subseteq (\lambda_1,\lambda_2)$, there exists $R' > 0$ with the following property: for every $\lambda \in A$ and, for every $G \in \mathcal{F}$ and every $q : V \to \mathbb{R}$ satisfy in the assumption (h_2) there exists $\delta > 0$ such that, for each $\nu \in [0,\delta]$, $\Phi'(u) - \lambda J'(u) - \mu\Psi'(u) = 0$ has at least three solutions in E. Hence, the problem $(P^{F,G}_{\lambda,\nu})$ has at least three weak solutions whose norms are less than R'. $\square$

4. Scalar case

As an application of the results from Section 3, we consider the problem

$$\begin{cases} \Delta u(x) + a(x)u(x) = \lambda p(x) f(u(x)) - \nu g(u(x)), \, x \in V/V_0 \\ u|_{V_0} = 0 \end{cases} \tag{$P^{f,g}_{\lambda,\nu}$}$$

where V, V_0, Δ, λ and ν are as introduced in the problem $(P^{F,G}_{\lambda,\nu})$ in Introduction. We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and that the variable potentials $a, p : V \to \mathbb{R}$ satisfy the conditions (h_1) and (h_2) in the Introduction.

Set $F(t) = \int_0^t f(\xi)\mathrm{d}\xi$ for all $t \in \mathbb{R}$. The following existence results are consequences of Theorem 3.4.

Theorem 4.1. *Assume that*

$(\mathcal{A}_6)$ *there exists a constant $\varepsilon > 0$ such that*

$$\sup_{x\in V} p(x). \max\left\{ \limsup_{u\to 0} \frac{F(u)}{u^2}, \limsup_{|u|\to\infty} \frac{F(u)}{u^2} \right\} < \varepsilon;$$

$(\mathcal{A}_7)$ *$a \neq 0$ and there exist a positive constant b such that*

$$\varepsilon < \frac{\Gamma(1 + N/2)F(b) \int_{B(x_0, \frac{D}{2})} p(x)\mathrm{d}\mu}{2b^2 D^{N-2}(2^N - 1)\pi^{N/2}(2N + 3)^2\|a\|_{\mathrm{L}^1(V)}}$$

where Γ is the Gamma function.

Then, for each compact interval $[c, d] \subset (\lambda_3, \lambda_4)$ where λ_3 and λ_4 are the same as λ_1 and λ_2, but $\int_V p(x)F(u_1(x), \dots, u_n(x))\mathrm{d}\mu$ replaced by $\int_V p(x)F(u(x))\mathrm{d}\mu$, respectively, there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every continuous function $g : \mathbb{R} \to \mathbb{R}$ there exists $\gamma > 0$ such that for each $\nu \in [0, \gamma]$, the problem $(P^{f,g}_{\lambda,\nu})$ has at least three weak solutions whose norms in E *are less than R.*

Theorem 4.2. *Assume that $a \neq 0$ and there exists a positive constant b such that $F(b) > 0$. Moreover, suppose that*

$$\limsup_{u\to 0} \frac{f(u)}{|u|} = \limsup_{|u|\to\infty} \frac{f(u)}{|u|} = 0. \tag{4.1}$$

Then, for each compact interval $[c, d] \subset (\lambda_3, \infty)$ where λ_3 is the same as λ_1 but $\int_V p(x)F(u_1(x), \dots, u_n(x))\mathrm{d}\mu$ replaced by $\int_V p(x)F(u(x))\mathrm{d}\mu$, there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every continuous function $g : \mathbb{R} \to \mathbb{R}$ there exists $\gamma > 0$ such that for each $\nu \in [0, \gamma]$, the problem $(P^{f,g}_{\lambda,\nu})$ has at least three weak solutions whose norms in E *are less than R.*

Proof. We easily observe that from (4.1) the assumption $(\mathcal{A}_6)$ is satisfied for every $\varepsilon > 0$. Moreover, using the assumptions $a \neq 0$ and $F(d) > 0$, by choosing $\varepsilon > 0$ small enough one can drive the assumption $(\mathcal{A}_7)$. Hence, the conclusion follows from Theorem 4.1. □

Remark 4.1. Our results show that no asymptotic conditions on f and g are required, and merely the algebraic conditions on f are supposed to guarantee the existence of solutions.

Remark 4.2. Our existence results to establish three solutions for the problem $(P^{f,g}_{\lambda,\nu})$ in Theorem 4.2 in the case $a = 0$ and the existence results of Breckner et al. in [10, Theorem 5.1] are the same. Indeed, the assumptions (4.1) in Theorem 4.2 and (C_2) in [10, Theorem 5.1] are the same.

Now, we present the following example to illustrate Theorem 4.2.

Example 4.1. Let

$$f(t) = \begin{cases} 2(t + \sin t)^2, & \text{if } t < \pi, \\ 2\pi^2 + \tanh(t - \pi), & \text{if } t \geq \pi. \end{cases}$$

Thus, $F(b) = F(1) = \int_0^1 2(t + \sin t)^2 \mathrm{d}t > 0$ and $\lim_{u\to 0} \frac{f(u)}{|u|} = \lim_{u\to\infty} \frac{f(u)}{|u|} = 0$. Hence, by applying Theorem 4.2 for each compact interval $[c, d] \subset (0, \infty)$, there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that, for each $\nu \in [0, \gamma]$, the problem

$$\begin{cases} \Delta u(x) - \frac{1}{1+x^2} u(x) = -\lambda e^x f(u(x)) - \nu g(u(x)), \; x \in V/V_0 \\ u|_{V_0} = 0 \end{cases}$$

has at least three weak solutions whose norms in the space E are less than R.

The following existence result is a consequences of Theorem 3.5.

Theorem 4.3. *Assume that there exist five positive constants b, c, α, κ and ξ with $2D^{N-2}\pi^{\frac{N}{2}}(2^N - 1)\|a\|_{\mathrm{L}^1(V)} b^2 < \Gamma(1 + N/2)(2N + 3)^2 c^2$ and $\alpha \in [0, 2)$ such that*

$(\mathcal{B}_4)$ $\max_{|u|\leq c} F(u) < \frac{F(b)}{\int_V p(x)\mathrm{d}\mu} \int_{B(x_0,\frac{D}{2})} p(x)\mathrm{d}\mu$;

$(\mathcal{B}_5)$ $F(u) > 0$ *for each* $u \in \mathbb{R}$;

$(\mathcal{B}_6)$ $|F(u)| \leq \kappa u^2 + \xi$ *for all* $u \in \mathbb{R}$.

Then, there exist a nonempty open set $A \subset [0, \infty)$ and a positive number R' with the following property: for every $\lambda \in A$ and every continuous function $g : \mathbb{R} \to \mathbb{R}$ there exists $\delta > 0$ such that, for each $\nu \in [0, \delta]$, the problem $(P^{f,g}_{\lambda,\nu})$ has at least three weak solutions whose norms in E *are less than R'.*

Remark 4.3. We should note that Theorem 4.3, under a weaker condition than Theorem 4.2, gives us the existence of three solutions for the problem $(P^{f,g}_{\lambda,\nu})$. Indeed, the assumption $\limsup_{|u|\to\infty} \frac{f(u)}{|u|} = 0$ in Theorem 4.3 is not necessary.

Finally, we present the following example to illustrate Theorem 4.3.

Example 4.2. Let $N = 2$, $x_0 = 0$, $a(x) = -\frac{1}{1+x^2}$ and $p(x) = -\frac{e^x}{4}$ for all $x \in V$, $f(t) = \frac{1}{1+t^2}$ for all $t \in \mathbb{R}$. Thus, f is a continuous function. By choosing $b = 1$, $c = 2$, $\alpha = 1$, $\kappa = 1$ and $\xi = \pi$ we have $\alpha = 1 \in [0, 2)$,

$$2D^{N-2}\pi^{\frac{N}{2}}(2^N - 1)\|a\|_{\mathrm{L}^1(V)} b^2 < 6\pi^2 < 196 = \Gamma(1 + N/2)(2N + 3)^2 c^2,$$

$$\max_{|u|\leq c} F(u) = \max_{|u|\leq 2} F(u) = \arctan(2) < \frac{\pi}{2} < \pi \leq \frac{F(b)}{\int_V p(x)\mathrm{d}\mu} \int_{B(x_0,\frac{D}{2})} p(x)\mathrm{d}\mu,$$

$$F(b) = F(1) = \arctan(1) = \frac{\pi}{4} > 0$$

and

$$|F(u)| \leq |u|^2 + \pi = \kappa u^2 + \xi \quad \text{for all } u \in \mathbb{R}.$$

Hence, by applying Theorem 4.3 and there exist a nonempty open set $A \subset [0, +\infty)$ and a positive number R' with the following property: for every $\lambda \in A$ and every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$ there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem

$$\begin{cases} \Delta u(x) - \frac{1}{1+x^2} u(x) = -\frac{\lambda e^x}{4} f(u(x)) - \nu g(u(x)), \, x \in V/V_0 \\ u|_{V_0} = 0 \end{cases}$$

has at least three weak solutions whose norms in the space E are less than R.

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