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ON THE CAGINALP PHASE-FIELD SYSTEM BASED ON TYPE III WITH TWO TEMPERATURES AND NONLINEAR COUPLING

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Abstract This paper is devoted to the study of a generalization of the Caginalp phase-field system based on the theory of type III thermomechanics with two temperatures for the heat conduction with a nonlinear coupling term. We start our analysis by establishing existence of the solutions. Then, we discuss dissipativity and uniqueness of the solutions. We finish our analysis by studying the spatial behavior of the solutions in a semi-infinite cylinder, assuming the existence of such solutions.

Keywords Caginalp system, nonlinear coupling, type III, spatial behaviour.

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1. Introduction

In recent years a new attention has been devoted to the Caginalp phase-field model [6]

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}. \tag{1.2}$$

Such a model describes the behavior of certain materials in their stages of melting and solidification. It has been instensively studied in several forms (see [1–5], [11–13, 15] for more details). In fact in this case θ and u can represent respectively the temperature and the order parameter.

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Using Fourier's law to the aforementioned model, one can observe a disparity between the observed results and the expected outcome. One of them is known as "paradox of heat conduction" and has been firstly underlined by Nernst (see [14]). In order to overcome this disparity several alternative laws have been proposed, the Maxwell-Cattaneo law (see [35]) or the Gurtin-Pipkin law (see [29] and [30]). The study of models derived from these new laws has been the subject of a number of papers regarding to the qualitative study of solutions (see [16, 21–25, 33, 39, 41–44]).

Furthermore, in [26–28] Green and Naghdi proposed an alternative theory based on a thermomechanical theory of deformable media to obtain very rational models. They considered three theories of thermoelasticity labeled as type I, II and III respectively (see [31, 32, 46] for more details).

We will focus on this latter type throughout this paper and in particular when the conductive temperature θ and the thermodynamic temperature T are differents. This case is shown to be more relevant for non-simple materials as described in [7–10, 37, 38, 45] and can be written

$$T = \theta - \Delta\theta. \tag{1.3}$$

We recall that for simple materials these temperatures are shown to coincide.

The purpose of our study is the following initial and boundary value problem

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g(u)\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right), \tag{1.4}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -g(u)\frac{\partial u}{\partial t}, \tag{1.5}$$

$$u|_{\partial\Omega} = \alpha|_{\partial\Omega} = 0, \tag{1.6}$$

$$u|_{t=0} = u_0, \quad \alpha|_{t=0} = \alpha_0, \quad \left.\frac{\partial \alpha}{\partial t}\right|_{t=0} = \alpha_1, \tag{1.7}$$

where Ω is a bounded and regular domain of $\mathbb{R}^n$ with $n = 2$ or 3.

This paper is divided as follows. The next section will be devoted to give a rigorous derivation of our model using type III and a nonlinear coupling. In Section 3 we will address the question of existence and regularity. Then we prove dissipativity and uniqueness of the solutions. We finish, in Section 5, by the study of the spatial behavior of the solutions in a semi-infinite cylinder, assuming that such solutions exist.

Throughout this paper, the same letters c, c' and c'' denote constants which may change from line to line and also $\|.\|_p$ will denote the usual L^p norm and $(.\,,\,.)$ the usual L^2 scalar product; more generally, we will denote by $\|.\|_X$ the norm in the Banach space X. When there is no possible confusion $\|.\|$ will be noted instead of $\|.\|_2$.

2. Derivation of the model

Our equations (1.4)-(1.7) modeling phase transition are derived as follows.

Let Ψ be the total energy of the system defined as

$$\Psi(u, T) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) - G(u)T - \frac{1}{2}T^2\right)dx,$$

with $G' = g$ and $F' = f$. The evolution equation for the order parameter is given by:

$$\frac{\partial u}{\partial t} = -\partial_u \Psi, \tag{2.1}$$

which yields owing to (1.3)

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = g(u)\big(\theta - \Delta\theta\big). \tag{2.2}$$

Let H be the enthalpy defined as follows

$$H = -\partial_T \Psi = G(u) + T = G(u) + \theta - \Delta\theta. \tag{2.3}$$

Furthermore,

$$\frac{\partial H}{\partial t} + \mathrm{divq} = 0. \tag{2.4}$$

Since

$$\frac{\partial H}{\partial t} = \frac{\partial \theta}{\partial t} - \Delta\frac{\partial \theta}{\partial t} + g(u)\frac{\partial u}{\partial t}. \tag{2.5}$$

In particular, considering the type III theory with two temperatures (see [45])

$$q = -\nabla\theta - \nabla\alpha, \tag{2.6}$$

where $\alpha(t,x) = \int_0^t \theta(\tau,x)d\tau + \alpha_0(x)$ is the conductive thermal displacement. Noting that $\theta = \frac{\partial \alpha}{\partial t}$. We get from (2.4) to (2.6),

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -g(u)\frac{\partial u}{\partial t}. \tag{2.7}$$

This leads to the above system (1.4)-(1.7).

3. Existence of solutions

We start by giving an existence result, the assumptions for the proof being the following: f is of class $\mathcal{C}^1$ and

$$|G(s)|^2 \le c_1\, F(s) + c_2, \quad c_0, c_1, c_2 \ge 0, \tag{3.1}$$

$$|g(s)s| \le c_3(|G(s)|^2 + 1), \quad c_3 \ge 0, \tag{3.2}$$

$$c_4\, s^{k+2} - c_5 \le F(s) \le f(s)s + c_0 \le c_6\, s^{k+2} - c_7, \quad c_4, c_6 > 0,\ c_5, c_7 \ge 0, \tag{3.3}$$

$$|g(s)| \le c_8(|s| + 1), \quad |g'(s)| \le c_9 \quad c_8, c_9 \ge 0, \tag{3.4}$$

$$|f'(s)| \le c_{10}(|s|^k + 1), \quad c_{10} \ge 0, \tag{3.5}$$

where k is an integer.

We have the

Theorem 3.1. *We assume that* (3.1)-(3.5) *hold true. If in addition* $(u_0, \alpha_0, \alpha_1) \in H_0^1(\Omega) \cap L^{k+2}(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega)$*, then* (1.4)-(1.7) *admits a solution* (u, α) *such that* $u \in L^\infty(0,T; H_0^1(\Omega) \cap L^{k+2}(\Omega))$, $\frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega))$, $\alpha \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega))$ *and* $\frac{\partial \alpha}{\partial t} \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega))$, $\forall T > 0$.

Proof. The proof of existence follows from the priori estimates and a standard Galerkin scheme (see [19, 20, 47] for details). We will just focus on the priori estimates.

Multiplying (1.4) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we have

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \frac{d}{dt}\int_\Omega F(u)dx = \int_\Omega g(u)\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)\frac{\partial u}{\partial t}dx. \tag{3.6}$$

Similarly, multiplying (1.5) by $\frac{\partial \alpha}{\partial t}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\alpha\|^2 = -\int_\Omega g(u)\frac{\partial u}{\partial t}\frac{\partial \alpha}{\partial t}dx. \tag{3.7}$$

Again, multiplying (1.5) by $-\Delta\frac{\partial \alpha}{\partial t}$, we get

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\Delta\alpha\|^2 = \int_\Omega g(u)\frac{\partial u}{\partial t}\Delta\frac{\partial \alpha}{\partial t}dx. \tag{3.8}$$

Summing (3.6), (3.7) and (3.8), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Bigg[\|\nabla u\|^2 + 2\int_\Omega F(u)dx + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 + \|\nabla\alpha\|^2\\ &+ \|\Delta\alpha\|^2\Bigg] + \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 = 0.\end{aligned} \tag{3.9}$$

Multiplying (1.4) by u, we find

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + (f(u), u) = \int_\Omega g(u)\left(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right)udx. \tag{3.10}$$

We have, owing to (3.1) and (3.10),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 + c_0\int_\Omega F(u)dx \leq & c\int_\Omega |G(u)|^2dx + \frac{1}{2}\left\|\frac{\partial \alpha}{\partial t}\right\|^2\\ &+ \frac{1}{2}\left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 + c'.\end{aligned} \tag{3.11}$$

From (3.9) and (3.11), we obtain

$$\begin{aligned}&\frac{d}{dt}E_1 + 2\|\nabla u\|^2 + 2c_0\int_\Omega F(u)dx + 2\left\|\frac{\partial u}{\partial t}\right\|^2 + 2\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + 2\left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2\\ &\leq c\int_\Omega |G(u)|^2dx + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial \alpha}{\partial t}\right\|^2 + c',\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}E_1 = &\|u\|^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)dx + \|\frac{\partial \alpha}{\partial t}\|^2 + 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2\\ &+ \|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\alpha\|^2 + \|\Delta\alpha\|^2\end{aligned} \tag{3.13}$$

satisfies (due to (3.3))

$$
\begin{aligned}
&E_1 \geq c\Big(\|u\|_{H^1(\Omega)}^2 + \|u\|_{k+2}^{k+2} + \|\alpha\|_{H^2(\Omega)}^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^2(\Omega)}^2\Big) - c', \\
&E_1 \leq c''\Big(\|u\|_{H^1(\Omega)}^2 + \|u\|_{k+2}^{k+2} + \|\alpha\|_{H^2(\Omega)}^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^2(\Omega)}^2\Big) - c'''.
\end{aligned} \tag{3.14}
$$

Multiplying (1.5) by α, we get

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{\partial \alpha}{\partial t}, \alpha\Big) + \frac{d}{dt}\Big(-\Delta\frac{\partial \alpha}{\partial t}, \alpha\Big) + \frac{1}{2}\frac{d}{dt}\|\nabla\alpha\|^2 + \|\nabla\alpha\|^2 \\
=&\Big\|\frac{\partial \alpha}{\partial t}\Big\|^2 + \Big\|\nabla\frac{\partial \alpha}{\partial t}\Big\|^2 - \int_\Omega g(u)\frac{\partial u}{\partial t}\alpha dx.
\end{aligned} \tag{3.15}
$$

Since

$$
\int_\Omega g(u)\frac{\partial u}{\partial t}\alpha dx = \frac{d}{dt}\int_\Omega G(u)\alpha dx - \int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx, \tag{3.16}
$$

(3.15) becomes

$$
\begin{aligned}
&\frac{d}{dt}\Big[\Big(\frac{\partial \alpha}{\partial t}, \alpha\Big) + \Big(-\Delta\frac{\partial \alpha}{\partial t}, \alpha\Big) + \int_\Omega G(u)\alpha dx + \frac{1}{2}\|\nabla\alpha\|^2\Big] + \|\nabla\alpha\|^2 \\
=&\Big\|\frac{\partial \alpha}{\partial t}\Big\|^2 + \Big\|\nabla\frac{\partial \alpha}{\partial t}\Big\|^2 + \int_\Omega G(u)\frac{\partial \alpha}{\partial t}dx.
\end{aligned} \tag{3.17}
$$

Adding (3.12) and η(3.17) with $\eta > 0$, we get

$$
\begin{aligned}
&\frac{d}{dt}E_2 + \|\nabla u\|^2 + 2c_0\int_\Omega F(u)dx + 2\Big\|\frac{\partial u}{\partial t}\Big\|^2 + 2\Big\|\nabla\frac{\partial \alpha}{\partial t}\Big\|^2 + 2\Big\|\Delta\frac{\partial \alpha}{\partial t}\Big\|^2 + \eta\|\nabla\alpha\|^2 \\
\leq& c\int_\Omega |G(u)|^2dx + (1+\eta)\Big\|\frac{\partial \alpha}{\partial t}\Big\|^2 + \eta\Big\|\nabla\frac{\partial \alpha}{\partial t}\Big\|^2 + \Big\|\Delta\frac{\partial \alpha}{\partial t}\Big\|^2 + c',
\end{aligned} \tag{3.18}
$$

with

$$
E_2 = E_1 + \eta\Big(\frac{\partial \alpha}{\partial t}, \alpha\Big) + \eta\Big(-\Delta\frac{\partial \alpha}{\partial t}, \alpha\Big) + \eta\int_\Omega G(u)\alpha dx + \frac{\eta}{2}\|\nabla\alpha\|^2.
$$

Hence

$$
\begin{aligned}
&\frac{d}{dt}E_2 + c\Big(\|u\|_{H^1(\Omega)}^2 + \int_\Omega F(u)dx + \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|_{H^2(\Omega)}^2 + \|\alpha\|_{H^1(\Omega)}^2\Big) \\
\leq& c'\int_\Omega |G(u)|^2dx + c''.
\end{aligned} \tag{3.19}
$$

Choosing η such that

$$
\begin{aligned}
&\Big\|\frac{\partial \alpha}{\partial t}\Big\|^2 + \eta\Big(\frac{\partial \alpha}{\partial t}, \alpha\Big) + \frac{1}{2}\|\nabla\alpha\|^2 \geq c\Big(\Big\|\frac{\partial \alpha}{\partial t}\Big\|^2 + \|\nabla\alpha\|^2\Big), \\
&2\Big\|\nabla\frac{\partial \alpha}{\partial t}\Big\|^2 + \eta\Big(-\Delta\frac{\partial \alpha}{\partial t}, \alpha\Big) + \frac{1}{2}\|\nabla\alpha\|^2 \geq c'\Big(\nabla\Big\|\frac{\partial \alpha}{\partial t}\Big\|^2 + \|\nabla\alpha\|^2\Big),
\end{aligned} \tag{3.20}
$$

and using (3.1), we have

$$2\int_\Omega F(u)dx + \frac{\eta}{2}\|\nabla\alpha\|^2 + \eta\int_\Omega G(u)\alpha dx \geq c\Big(\int_\Omega F(u)dx + \|\nabla\alpha\|^2\Big) - \eta\, c_2. \quad (3.21)$$

It follows from (3.14), (3.20) and (3.21),

$$E_2 \leq c\Big(\|u\|^2_{H^1(\Omega)} + \|u\|^{k+2}_{k+2} + \Big\|\frac{\partial\alpha}{\partial t}\Big\|^2_{H^2(\Omega)} + \|\alpha\|^2_{H^2(\Omega)}\Big) + k_1, \quad k_1 > 0. \quad (3.22)$$

Similarly

$$E_2 \geq c\Big(\|u\|^2_{H^1(\Omega)} + \|u\|^{k+2}_{k+2} + \Big\|\frac{\partial\alpha}{\partial t}\Big\|^2_{H^2(\Omega)} + \|\alpha\|^2_{H^2(\Omega)}\Big) - k_2, \quad k_2 > 0. \quad (3.23)$$

We deduce owing to (3.1) and (3.19)

$$\frac{d}{dt}E_2 + c\Big\|\frac{\partial u}{\partial t}\Big\|^2_2 \leq c'\, E_2 + c''. \quad (3.24)$$

Finally the proof is deduced from (3.22)-(3.24). □

Now, we make the following assumption

$$f(0) = 0, \quad f' \geq -k_3, \quad k_3 \geq 0. \quad (3.25)$$

We also have the following theorem with more regularity

Theorem 3.2. *We assume that the assumptions of Theorem 3.1 are satisfied and also* (3.25) *hold true. If in addition* $(u_0, \alpha_0, \alpha_1) \in H^1_0(\Omega) \cap L^{k+2}(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega)$*, then* (1.4)-(1.7) *admits a solution* (u, α) *such that* $u \in L^\infty(0, T; H^1_0(\Omega) \cap L^{k+2}(\Omega))$, $\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$, $\alpha \in L^\infty(0, T; H^1_0(\Omega))$ *and* $\frac{\partial\alpha}{\partial t} \in L^\infty(0, T; H^1_0(\Omega) \cap H^2(\Omega))$, $\forall T > 0$.

Proof. As in a previous theorem we give the priori estimates on the solutions.

Multiplying (1.5) by $-\Delta\alpha$, integrating over Ω and by parts, we have

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{\partial\alpha}{\partial t}, -\Delta\alpha\Big) + \frac{d}{dt}\Big(\Delta\frac{\partial\alpha}{\partial t}, \Delta\alpha\Big) + \frac{1}{2}\frac{d}{dt}\|\Delta\alpha\|^2 + \|\Delta\alpha\|^2\\ =&\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2 + \Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2 + \int_\Omega g(u)\frac{\partial u}{\partial t}\Delta\alpha dx.\end{aligned} \quad (3.26)$$

Since

$$\int_\Omega g(u)\frac{\partial u}{\partial t}\Delta\alpha dx = \frac{d}{dt}\int_\Omega G(u)\Delta\alpha dx - \int_\Omega G(u)\Delta\frac{\partial\alpha}{\partial t}dx, \quad (3.27)$$

it follows from (3.26)

$$\begin{aligned}&\frac{d}{dt}\Big[\Big(\frac{\partial\alpha}{\partial t}, -\Delta\alpha\Big) + \Big(\Delta\frac{\partial\alpha}{\partial t}, \Delta\alpha\Big) - \int_\Omega G(u)\Delta\alpha dx + \frac{1}{2}\|\Delta\alpha\|^2\Big] + \|\Delta\alpha\|^2\\ \leq& c\Big(\int_\Omega |G(u)|^2 dx + \Big\|\frac{\partial\alpha}{\partial t}\Big\|^2_{H^2(\Omega)}\Big).\end{aligned} \quad (3.28)$$

We differentiate (1.4) with respect to time to get

$$\frac{\partial^2 u}{\partial t^2} - \Delta\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} = g'(u)\frac{\partial u}{\partial t}\Big(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\Big) + g(u)\Big(\frac{\partial^2 u}{\partial t^2} - \Delta\frac{\partial^2 u}{\partial t^2}\Big). \tag{3.29}$$

Using (1.5), we obtain

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} - \Delta\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial t} =& g'(u)\frac{\partial u}{\partial t}\Big(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\Big)\\ &+ g(u)\Big(\Delta\frac{\partial \alpha}{\partial t} + \Delta\alpha - g(u)\frac{\partial u}{\partial t}\Big).\end{aligned} \tag{3.30}$$

Multiplying (3.30) by $\frac{\partial u}{\partial t}$, we have owing to (3.25) and the boundary condition $\frac{\partial u}{\partial t} = 0 \quad \text{on} \quad \partial\Omega$,

$$\begin{aligned}\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} \leq& k_3\|\frac{\partial u}{\partial t}\|^2 + \int_\Omega g'(u)|\frac{\partial u}{\partial t}|^2\Big(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\Big)dx\\ &+ \int_\Omega g(u)\Delta\frac{\partial \alpha}{\partial t}\frac{\partial u}{\partial t}dx + \int_\Omega g(u)\Delta\alpha\frac{\partial u}{\partial t}dx\\ &- \int_\Omega (g(u))^2|\frac{\partial u}{\partial t}|^2 dx.\end{aligned} \tag{3.31}$$

Let us start by giving estimates of the right-hand side of the inequality (3.31). We have owing to (3.4) and classical Sobolev embeddings

$$\begin{aligned}\Big|\int_\Omega g'(u)|\frac{\partial u}{\partial t}|^2\Big(\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\Big)dx\Big| &\leq c_9\int_\Omega |\frac{\partial u}{\partial t}|^2\Big|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\Big|dx\\ &\leq c\Big(\|\frac{\partial u}{\partial t}\|^4_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2\\ &\qquad + \|\Delta\frac{\partial \alpha}{\partial t}\|^2\Big).\end{aligned} \tag{3.32}$$

We note that

$$\int_\Omega g(u)\Delta\frac{\partial \alpha}{\partial t}\frac{\partial u}{\partial t}dx = -\int_\Omega g'(u)\frac{\partial u}{\partial t}\nabla\frac{\partial \alpha}{\partial t}\nabla u dx - \int_\Omega g(u)\nabla\frac{\partial \alpha}{\partial t}\nabla\frac{\partial u}{\partial t}dx, \tag{3.33}$$

and also

$$\int_\Omega g(u)\Delta\alpha\frac{\partial u}{\partial t}dx = -\int_\Omega g'(u)\frac{\partial u}{\partial t}\nabla\alpha\nabla u dx - \int_\Omega g(u)\nabla\alpha\nabla\frac{\partial u}{\partial t}dx. \tag{3.34}$$

In addition

$$\begin{aligned}\Big|\int_\Omega g'(u)\frac{\partial u}{\partial t}\nabla\frac{\partial \alpha}{\partial t}\nabla u dx\Big| &\leq c_9\int_\Omega |\frac{\partial u}{\partial t}||\nabla\frac{\partial \alpha}{\partial t}||\nabla u|dx\\ &\leq c\Big(\|\nabla\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}\|\nabla u\|^2\Big),\end{aligned} \tag{3.35}$$

and also

$$\begin{aligned}\Big|\int_\Omega g(u)\nabla\frac{\partial u}{\partial t}\nabla\frac{\partial \alpha}{\partial t}dx\Big| &\leq c_8\int_\Omega (|u|+1)|\nabla\frac{\partial u}{\partial t}||\nabla\frac{\partial \alpha}{\partial t}|dx\\ &\leq c\Big[\|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)}(\|\nabla u\|^2 + 1)\Big].\end{aligned} \tag{3.36}$$

Similarly

$$\left| \int_\Omega g'(u) \frac{\partial u}{\partial t} \nabla \frac{\partial \alpha}{\partial t} \nabla u dx \right| \leq c\Big(\|\nabla \frac{\partial u}{\partial t}\|^2 + \|\alpha\|^2_{H^2(\Omega)} \|\nabla u\|^2 \Big), \tag{3.37}$$

and also

$$\left| \int_\Omega g(u) \nabla \frac{\partial u}{\partial t} \nabla \frac{\partial \alpha}{\partial t} dx \right| \leq c\Big[\|\frac{\partial u}{\partial t}\|^2 + \|\alpha\|^2_{H^2(\Omega)} \big(\|\nabla u\|^2 + 1\big) \Big]. \tag{3.38}$$

Furthermore

$$\begin{aligned} \int_\Omega (g(u))^2 |\frac{\partial u}{\partial t}|^2 dx &\leq c \int_\Omega (|u|+1)^2 |\frac{\partial u}{\partial t}|^2 dx \\ &\leq c' \Big(\|u\|^2_{H^1(\Omega)} + \|u\| + 1 \Big) \|\frac{\partial u}{\partial t}\|^2. \end{aligned} \tag{3.39}$$

It follows from (3.32)-(3.39) and (3.31) that

$$\begin{aligned} \frac{d}{dt} \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} \leq c\Big(&\|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)} \big(\|\nabla u\|^2 + 1\big) \\ &+ \|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} \big(\|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} + \|u\| + 1\big) \Big). \end{aligned} \tag{3.40}$$

Summing (3.18), $\eta_1 \times$(3.28) and $\eta_2 \times$(3.40), where $\eta_1,\eta_2 > 0$ are small enough, we deduce owing to (3.22) and (3.23)

$$\begin{aligned} &\frac{d}{dt} E_3 + c\Big(\|u\|^2_{H^1(\Omega)} + \int_\Omega F(u) dx + \|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)} + \|\alpha\|^2_{H^2(\Omega)} \Big) \\ &\leq c' \int_\Omega |G(u)|^2 dx + c'', \end{aligned} \tag{3.41}$$

where

$$\begin{aligned} E_3 =& E_2 + \eta_1 \Big(\frac{\partial \alpha}{\partial t}, -\Delta\alpha \Big) + \eta_1 \Big(\Delta \frac{\partial \alpha}{\partial t}, \Delta\alpha \Big) - \eta_1 \int_\Omega G(u) \Delta\alpha dx \\ &+ \frac{\eta_1}{2} \|\Delta\alpha\|^2 + \eta_2 \|\frac{\partial u}{\partial t}\|^2 \end{aligned} \tag{3.42}$$

satisfies (owing to (3.22) and (3.23))

$$\begin{aligned} E_3 &\geq c\Big(\|u\|^2_{H^1(\Omega)} + \|u\|^{k+2}_{k+2} + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)} \Big) - c', \\ E_3 &\leq c''\Big(\|u\|^2_{H^1(\Omega)} + \|u\|^{k+2}_{k+2} + \|\alpha\|^2_{H^2(\Omega)} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^2(\Omega)} \Big) - c'''. \end{aligned} \tag{3.43}$$

This leads us to

$$\frac{d}{dt} E_3 + c \left\| \frac{\partial u}{\partial t} \right\|^2_{H^1(\Omega)} \leq c' \, E_3 + c'', \tag{3.44}$$

The proof follows from Gronwall's lemma. □

Remark 3.1. Assumption (3.25) is reasonable and should be a physically realistic choice especially when F can be taken as a double well potential of degree 4. We can take for example $F(u) = \frac{1}{4}\big(u^2 - 1\big)^2$ for which $k_3 = 1$.

4. Dissipativity and uniqueness of the system

Let us consider a more restrictive assumption on G as follows:

$$\forall \epsilon > 0, \quad |G(s)|^2 \leq \epsilon F(s) + c_\epsilon, \quad s \in \mathbb{R}. \tag{4.1}$$

We also have the

Theorem 4.1. *We assume that* (3.2)-(3.4) *hold true and*

$$(u_0, \alpha_0, \alpha_1) \in H_0^1(\Omega) \cap L^{k+2}(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \cap H^2(\Omega).$$

If in addition (4.1) *is satisfied, then* $u \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)) \cap L^\infty(\mathbb{R}^+; L^{k+2}(\Omega))$, $\alpha \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)) \cap L^\infty(\mathbb{R}^+; H^2(\Omega))$ *and* $\frac{\partial \alpha}{\partial t} \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)) \cap L^\infty(\mathbb{R}^+; H^2(\Omega))$.

Proof. We get after computation

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + f(u_1) - f(u_2) =& g(u_1)\Big[\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\Big] \\ &+ \big(g(u_1) - g(u_2)\big)\Big[\frac{\partial \alpha_2}{\partial t} - \Delta \frac{\partial \alpha_2}{\partial t}\Big], \end{aligned} \tag{4.2}$$

and

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -g(u_1)\frac{\partial u}{\partial t} - \big(g(u_1) - g(u_2)\big)\frac{\partial u_2}{\partial t}, \tag{4.3}$$

where $u = u_1 - u_2$ and $\alpha = \alpha_1 - \alpha_2$.

We multiply (4.2) by $\frac{\partial u}{\partial t}$ and (4.2) by $\left(\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\right)$. After integrating over Ω, we then obtain

$$\begin{aligned} &\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \frac{1}{2}\frac{d}{dt}\big\|\nabla u\big\|^2 + \int_\Omega \big(f(u_1) - f(u_2)\big)\frac{\partial u}{\partial t}dx \\ &= \int_\Omega g(u_1)\Big[\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\Big]\frac{\partial u}{\partial t}dx \int_\Omega \big(g(u_1) - g(u_2)\big)\Big[\frac{\partial \alpha_2}{\partial t} - \Delta \frac{\partial \alpha_2}{\partial t}\Big]\frac{\partial u}{\partial t}dx, \end{aligned} \tag{4.4}$$

and also

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\|\frac{\partial \alpha}{\partial t}\Big\|^2 + \frac{d}{dt}\Big\|\nabla \frac{\partial \alpha}{\partial t}\Big\|^2 + \frac{1}{2}\frac{d}{dt}\Big\|\Delta \frac{\partial \alpha}{\partial t}\Big\|^2 + \frac{1}{2}\frac{d}{dt}\big\|\nabla \alpha\big\|^2 + \frac{1}{2}\frac{d}{dt}\big\|\Delta \alpha\big\|^2 \\ &\quad + \Big\|\nabla \frac{\partial \alpha}{\partial t}\Big\|^2 + \Big\|\Delta \frac{\partial \alpha}{\partial t}\Big\|^2 \\ &= -\int_\Omega g(u_1)\Big[\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\Big]\frac{\partial u}{\partial t}dx + \int_\Omega \big(g(u_1) - g(u_2)\big)\frac{\partial u_2}{\partial t}\Big[\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\Big]dx. \end{aligned} \tag{4.5}$$

Adding (4.4) and (4.5), we get

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{1}{2}\big\|\nabla u\big\|^2 + \frac{1}{2}\Big\|\frac{\partial \alpha}{\partial t}\Big\|^2 + \Big\|\nabla \frac{\partial \alpha}{\partial t}\Big\|^2 + \frac{1}{2}\Big\|\Delta \frac{\partial \alpha}{\partial t}\Big\|^2 + \frac{1}{2}\big\|\nabla \alpha\big\|^2 + \frac{1}{2}\big\|\Delta \alpha\big\|^2\Big] \\ &\quad + \Big\|\nabla \frac{\partial \alpha}{\partial t}\Big\|^2 + \Big\|\Delta \frac{\partial \alpha}{\partial t}\Big\|^2 \\ &\leq \int_\Omega \big|f(u_1) - f(u_2)\big|\Big|\frac{\partial u}{\partial t}\Big|dx + \int_\Omega \big|g(u_1) - g(u_2)\big|\Big|\frac{\partial \alpha_2}{\partial t} - \Delta \frac{\partial \alpha_2}{\partial t}\Big|\Big|\frac{\partial u}{\partial t}\Big|dx \\ &\quad + \int_\Omega \big|g(u_1) - g(u_2)\big|\Big|\frac{\partial u_2}{\partial t}\Big|\Big|\frac{\partial \alpha}{\partial t} - \Delta \frac{\partial \alpha}{\partial t}\Big|dx. \end{aligned} \tag{4.6}$$

We have, using (3.4),(3.5) and noting that $n = 3$ when $k = 2$,

$$\begin{aligned}\int_\Omega |f(u_1) - f(u_2)| \left|\frac{\partial u}{\partial t}\right| dx &\le c_{10} \int_\Omega (|u_2|^k + 1)\,|u|\,\left|\frac{\partial u}{\partial t}\right| dx \\ &\le c(\|u_2\|^{2k}_{H^1(\Omega)} + 1)\|\nabla u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2,\end{aligned} \tag{4.7}$$

$$\begin{aligned}\int_\Omega |g(u_1) - g(u_2)|\left|\frac{\partial \alpha_2}{\partial t} - \Delta\frac{\partial \alpha_2}{\partial t}\right|\left|\frac{\partial u}{\partial t}\right| dx &\le c_9 \int_\Omega |u|\left|\frac{\partial \alpha_2}{\partial t} - \Delta\frac{\partial \alpha_2}{\partial t}\right|\left|\frac{\partial u}{\partial t}\right| dx \\ &\le c\Big(\|\frac{\partial \alpha_2}{\partial t} - \Delta\frac{\partial \alpha_2}{\partial t}\|^2\|u\|^2_{H^1(\Omega)}\Big) \\ &\quad + \left\|\frac{\partial u}{\partial t}\right\|^2_{H^1(\Omega)}\end{aligned} \tag{4.8}$$

and also

$$\begin{aligned}\int_\Omega |g(u_1) - g(u_2)|\left|\frac{\partial u_2}{\partial t}\right|\left|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right| dx &\le c_9 \int_\Omega |u|\left|\frac{\partial u_2}{\partial t}\right|\left|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\right| dx \\ &\le c\Big(\|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2\|\frac{\partial \alpha_2}{\partial t}\|^2_{H^1(\Omega)}\Big) \\ &\quad + \|u\|^2_{H^1(\Omega)}.\end{aligned} \tag{4.9}$$

Taking estimates (4.7)-(4.9) into account, we can see that from (4.6) we deduce the following differential inequality

$$\frac{dE_4}{dt} \le cJ\Big(\|\frac{\partial \alpha}{\partial t} - \Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\frac{\partial u}{\partial t}\|^2_{H^1(\Omega)}\Big), \tag{4.10}$$

where

$$E_4 = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \frac{1}{2}\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \frac{1}{2}\|\nabla\alpha\|^2 + \frac{1}{2}\|\Delta\alpha\|^2$$

and

$$J = J\Big(\|\frac{\partial \alpha_2}{\partial t}\|^2, \|\nabla\frac{\partial \alpha_2}{\partial t}\|^2, \|\Delta\frac{\partial \alpha_2}{\partial t}\|^2\Big).$$

Therefore, using the Gronwall's lemma, we obtain the uniqueness of the solution. This ends the proof. $\square$

Remark 4.1. It follows from Theorem 3.2 and Theorem 4.1 that we can define by setting $\Phi = L^{k+2}(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega)$ a semigroup

$$\begin{aligned} S(t): \quad & \Phi \to \Phi \\ & u_0 \to u(t), \quad t \ge 0 \end{aligned} \tag{4.11}$$

such that $S(t)$ is dissipative in Φ. This means that it possesses a bounded absorbing set $B_0 \subset \Phi$ (i.e, $\forall B \subset \Phi$ bounded , $\exists t_0 = t_0(B)$ such that $t_0 \ge t_0$ implies $S(t)B \subset B_0$). We refer the reader to [13, 18, 34, 40, 48] for more details about this subject.

5. Spatial behavior of the solutions

5.1. Assumptions

To study the spatial behavior of the solutions in a semi-infinite cylinder we need to add some assumptions. We first assume that such solutions exist. We then consider the boundary conditions

$$u = \alpha = 0 \quad \text{on} \quad (0, +\infty) \times \partial \mathrm{D} \times (0, \mathrm{T}), \tag{5.1}$$

$$u(0, x_2, x_3, t) = h_1(x_2, x_3, t), \tag{5.2}$$

$$\alpha(0, x_2, x_3, t) = h_2(x_2, x_3, t) \quad \text{on} \quad \{0\} \times \mathrm{D} \times (0, \mathrm{T}) \tag{5.3}$$

and the initial conditions

$$u|_{t=0} = \alpha|_{t=0} = \frac{\partial \alpha}{\partial t}\Big|_{t=0} = 0 \quad \text{on} \quad \mathrm{R}. \tag{5.4}$$

Here D denotes a two dimensional bounded domain and R a semi-infinite cylinder $(0, +\infty) \times D$. We consider the function

$$F_\omega(z, t) = \int_0^t \int_{D(z)} e^{-(w\, s)} [u_s u_{,1} + \alpha_s(\alpha_{,1} + \alpha_{,1s} + \alpha_{,1ss}) + \alpha_{ss}\alpha_{,1s}] da\, ds, \tag{5.5}$$

$$G_\omega(z,t) = \int_0^t \int_{D(z)} e^{-w\, s} \Big(uu_{,1} + \alpha(\alpha_{,1} + \alpha_{,1s} + \theta_{,1}) + \alpha_{ss}(\alpha_{,1} + \alpha_{,1s} + \alpha_{,1ss}) \Big) da\, ds \tag{5.6}$$

and $H_\omega = F_\omega + \tau G_\omega$, where $D(z) = \{x \in R, \quad x_1 = z\}$, $u_{,1} = \frac{\partial u}{\partial x_1}$, $u_s = \frac{\partial u}{\partial s}$, $\theta(t) = \int_0^t \alpha(s) ds$, w is a positive constant and τ is choosing large enough.

We will sometimes use some assumptions on the functions F and G; these will be specified later on.

5.2. Estimates on F_ω, G_ω and H_ω

By differentiating the term F_ω, we get using the divergence theorem

$$\begin{aligned}
&F_\omega(z + h, t) - F_\omega(z, t) \\
=& \frac{e^{-\omega t}}{2} \int_0^t \int_{R(z,z+h)} \Big(|\nabla u|^2 + 2F(u) + |\nabla \alpha|^2 + |\Delta \alpha|^2 + |\alpha_s|^2 + 2|\nabla \alpha_s|^2 + |\Delta \alpha_s|^2 \Big) da \\
&+ \int_0^t \int_{R(z,z+h)} e^{-\omega\, s} \Big(|u_s|^2 + |\nabla \alpha_s|^2 + |\Delta \alpha_s|^2 \Big) da\, ds \\
&+ \frac{\omega}{2} \int_0^t \int_{R(z,z+h)} e^{-\omega s} \Big(|\nabla u|^2 + 2F(u) + |\nabla \alpha|^2 + |\Delta \alpha|^2 + |\alpha_s|^2 + 2|\nabla \alpha_s|^2 + |\Delta \alpha_s|^2 \Big) dads
\end{aligned} \tag{5.7}$$

where $R(z, z+h) = \{x \in R, z < x_1 < z+h\}$. Hence

$$\begin{aligned}\frac{\partial F_\omega(z,t)}{\partial z} =& \frac{e^{-\omega t}}{2}\int_0^t\int_{D(z)}\Big(|\nabla u|^2+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2+|\alpha_s|^2+2|\nabla\alpha_s|^2+|\Delta\alpha_s|^2\Big)da\\ &+\int_0^t\int_{D(z)} e^{-\omega s}\Big(|u_s|^2+|\nabla\alpha_s|^2+|\Delta\alpha_s|^2\Big)da\,ds\\ &+\frac{\omega}{2}\int_0^t\int_{D(z)} e^{-\omega s}\Big(|\nabla u|^2+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2+|\alpha_s|^2+2|\nabla\alpha_s|^2\\ &+|\Delta\alpha_s|^2\Big)da\,ds. \end{aligned} \tag{5.8}$$

Similarly

$$\begin{aligned} &G_\omega(z+h,t)-G_\omega(z,t)\\ =&\frac{e^{-\omega t}}{2}\int_{R(z,z+h)}\Big(|u|^2+|\alpha|^2+(1+\omega)|\nabla\alpha|^2+|\nabla\alpha_s|^2+|\nabla\theta|^2+2\nabla\alpha\nabla\alpha_s\Big)dx\\ &+\int_0^t\int_{R(z,z+h)} e^{-\omega s}\Big(|\nabla u|^2+|\nabla\alpha|^2+|\alpha_{ss}|^2+|\nabla\alpha_{ss}|^2-|\nabla\alpha_s|^2\Big)dx\,ds\\ &+\frac{\omega}{2}\int_0^t\int_{R(z,z+h)} e^{-\omega s}\Big(|u|^2+|\alpha|^2+(1+\omega)|\nabla\alpha|^2+|\nabla\alpha_s|^2+|\nabla\theta|^2\Big)dx\,ds\\ &+\int_0^t\int_{R(z,z+h)} e^{-\omega s}\Big(f(u)u+g(u)u_s\alpha_{ss}+G(u)\alpha-g(u)u(\alpha_s-\Delta\alpha_s)\Big)dx\,ds. \end{aligned} \tag{5.9}$$

This leads us to

$$\begin{aligned}\frac{\partial G_\omega(z,t)}{\partial z} =& \frac{e^{-\omega t}}{2}\int_{D(z)}\Big(|u|^2+|\alpha|^2+(1+\omega)|\nabla\alpha|^2+|\nabla\alpha_s|^2+|\nabla\theta|^2+2\nabla\alpha\nabla\alpha_s\Big)da\\ &+\int_0^t\int_{D(z)} e^{-\omega s}\Big(|\nabla u|^2+|\nabla\alpha|^2+|\alpha_{ss}|^2+|\nabla\alpha_{ss}|^2-|\nabla\alpha_s|^2\Big)da\,ds\\ &+\frac{\omega}{2}\int_0^t\int_{D(z)} e^{-\omega s}\Big(|u|^2+|\alpha|^2+(1+\omega)|\nabla\alpha|^2+|\nabla\alpha_s|^2+|\nabla\theta|^2\Big)da\,ds\\ &+\int_0^t\int_{D(z)} e^{-\omega s}\Big(f(u)u+g(u)u_s\alpha_{ss}+G(u)\alpha-g(u)u(\alpha_s-\Delta\alpha_s)\Big)da\,ds. \end{aligned} \tag{5.10}$$

We now make the following assumptions on the nonlinear terms f, F, g, and G :

$$2F(u)+\tau|u|^2 \geq C_1(|u|^2+|u|^k), \quad f(u)u+\frac{\omega}{2}|u|^2 \geq C_2|u|^2 \tag{5.11}$$

and

$$\begin{aligned} &|u_s|^2+|\Delta\alpha_s|^2+\frac{\omega}{2}|\alpha_s|^2+\tau\frac{\omega}{2}(|u|^2+|\alpha|^2)\\ &+\tau\Big(g(u)u_s\alpha_{ss}+G(u)\alpha-g(u)u(\alpha_s-\Delta\alpha_s)\Big)\\ \geq& C_3(|u|^2+|\alpha|^2+|u_s|^2+|\alpha_s|^2+|\Delta\alpha_s|^2+|\alpha_{ss}|^2), \end{aligned} \tag{5.12}$$

where C_1, C_2 and C_3 are positive constants.
It follows from (5.11), (5.12) and choosing ω and τ large enough

$$\begin{aligned}
&|\nabla u|^2+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2+|\alpha_s|^2+2|\nabla\alpha_s|^2+|\Delta\alpha_s|^2\\
&+\tau\big(|u|^2+|\alpha|^2+(1+\omega)|\nabla\alpha|^2+|\nabla\alpha_s|^2+|\nabla\theta|^2+2\nabla\alpha\nabla\alpha_s\big)\\
\geq& K_1\bigg(|u|^2+|\nabla u|^2+|\alpha|^2+|\nabla\alpha|^2+|\Delta\alpha|^2+|\alpha_s|^2+|\nabla\alpha_s|^2+|\Delta\alpha_s|^2+|\nabla\theta|^2\bigg)
\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}
&|u_s|^2+|\nabla\alpha_s|^2+|\Delta\alpha_s|^2+\tau\bigg(|\nabla u|^2+|\nabla\alpha|^2+|\alpha_{ss}|^2+|\nabla\alpha_{ss}|^2-|\nabla\alpha_s|^2\bigg)\\
&+\frac{\omega}{2}\bigg(|\nabla u|^2+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2+|\alpha_s|^2+2|\nabla\alpha_s|^2\\
&+|\Delta\alpha_s|^2+\tau\big(|u|^2+|\alpha|^2+(1+\omega)|\nabla\alpha|^2+|\nabla\alpha_s|^2+|\nabla\theta|^2\big)\bigg)\\
&+\tau\bigg(f(u)u+g(u)u_s\alpha_{ss}+G(u)\alpha-g(u)u(\alpha_s-\Delta\alpha_s)\bigg)\\
\geq& K_2\bigg(|u|^2+|\nabla u|^2+|\alpha|^2+|\nabla\alpha|^2+|\Delta\alpha|^2+|u_s|^2+|\alpha_s|^2+|\nabla\alpha_s|^2\\
&+|\Delta\alpha_s|^2+|\alpha_{ss}|^2+|\nabla\alpha_{ss}|^2+|\nabla\theta|^2\bigg)
\end{aligned} \tag{5.14}$$

where K_1 and K_2 are positive constants. Remembering that $\frac{\partial H_\omega}{\partial z}=\frac{\partial F_\omega}{\partial z}+\tau\frac{\partial G_\omega}{\partial z}$, we deduce from (5.8), (5.10), (5.13) and (5.14)

$$\begin{aligned}
\frac{\partial H_\omega(z,t)}{\partial z}\geq& K_3\int_0^t\int_{D(z)}\bigg(|u|^2+|\nabla u|^2+|\alpha|^2+|\nabla\alpha|^2+|\Delta\alpha|^2+|u_s|^2\\
&+|\alpha_s|^2+2|\nabla\alpha_s|^2+|\Delta\alpha_s|^2+|\alpha_{ss}|^2+|\nabla\alpha_{ss}|^2+|\nabla\theta|^2\bigg)da ds
\end{aligned} \tag{5.15}$$

with $K_3>0$.

Now, we estimate $|H_\omega|$ in terms of $\frac{\partial H_\omega}{\partial z}$.

Applying Cauchy-Schwarz's inequality, one has

$$\begin{aligned}
|F_\omega|\leq&\bigg(\int_0^t\int_{D(z)}e^{-(\omega s)}u_s^2da\,ds\bigg)^{1/2}\bigg(\int_0^t\int_{D(z)}e^{-(\omega s)}u_{,1}^2da\,ds\bigg)^{1/2}\\
&+\bigg(\int_0^t\int_{D(z)}e^{-(\omega s)}\alpha_s^2da\,ds\bigg)^{1/2}\bigg(\int_0^t\int_{D(z)}e^{-(\omega s)}\alpha_{,1}^2da\,ds\bigg)^{1/2}\\
&+\bigg(\int_0^t\int_{D(z)}e^{-(\omega s)}\alpha_s^2da\,ds\bigg)^{1/2}\bigg(\int_0^t\int_{D(z)}e^{-(\omega s)}\alpha_{,1s}^2da\,ds\bigg)^{1/2}\\
&+\bigg(\int_0^t\int_{D(z)}e^{-(\omega s)}\alpha_s^2da\,ds\bigg)^{1/2}\bigg(\int_0^t\int_{D(z)}e^{-(\omega s)}\alpha_{,1ss}^2da\,ds\bigg)^{1/2}\\
&+\bigg(\int_0^t\int_{D(z)}e^{-(\omega s)}\alpha_{ss}^2da\,ds\bigg)^{1/2}\bigg(\int_0^t\int_{D(z)}e^{-(\omega s)}\alpha_{,1s}^2da\,ds\bigg)^{1/2}
\end{aligned}$$

$$\leq K_4 \int_0^t \int_{D(z)} e^{-\omega s} \Big(|\nabla u|^2 + |u_s|^2 + |\nabla \alpha|^2 + |\alpha_s|^2 + |\nabla \alpha_s|^2 + |\alpha_{ss}|^2 + |\nabla \alpha_{ss}|^2 \Big) da\, ds. \tag{5.16}$$

Similarly,

$$\begin{aligned}
|G_\omega| \leq & \left(\int_0^t \int_{D(z)} e^{-\omega s} u^2 da\, ds \right)^{1/2} \left(\int_0^t \int_{D(z)} e^{-\omega s} u_{,1}^2 da\, ds \right)^{1/2} \\
& + \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha^2 da\, ds \right)^{1/2} \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha_{,1}^2 da\, ds \right)^{1/2} \\
& + \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha^2 da\, ds \right)^{1/2} \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha_{,1s}^2 da\, ds \right)^{1/2} \\
& + \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha^2 da\, ds \right)^{1/2} \left(\int_0^t \int_{D(z)} e^{-\omega s} \theta_{,1}^2 da\, ds \right)^{1/2} \\
& + \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha_{ss}^2 da\, ds \right)^{1/2} \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha_{,1}^2 da\, ds \right)^{1/2} \\
& + \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha_{ss}^2 da\, ds \right)^{1/2} \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha_{,1s}^2 da\, ds \right)^{1/2} \\
& + \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha_{ss}^2 da\, ds \right)^{1/2} \left(\int_0^t \int_{D(z)} e^{-\omega s} \alpha_{,1ss}^2 da\, ds \right)^{1/2} \\
\leq & K_5 \int_0^t \int_{D(z)} e^{-\omega s} \Big(|u|^2 + |\nabla u|^2 + |\alpha|^2 + |\nabla \alpha|^2 + |\nabla \alpha_s|^2 \\
& + |\alpha_{ss}|^2 + |\nabla \alpha_{ss}|^2 + |\nabla \theta|^2 \Big) da\, ds, K_5 > 0.
\end{aligned} \tag{5.17}$$

We finally deduce from (5.15), (5.16) and (5.17) the existence of $K > 0$ such that

$$|H_\omega| \leq K \frac{\partial H_\omega}{\partial z}, \tag{5.18}$$

where $K = \frac{K_4 + \tau K_5}{K_3}$.

Remark 5.1. The inequality (5.18) is well known in the framework of spatial behavior as the Phragmén-Lindelöf alternative (see [18], [42] and references therein).

We give in what follows the main result of this section

5.3. Main result

We set

$$\begin{aligned}
\mathcal{E}_\omega(z,t) = & \frac{1}{2} \int_{R(z)} \Big(|\nabla u|^2 + 2F(u) + |\nabla \alpha|^2 + |\Delta \alpha|^2 + |\alpha_s|^2 + 2|\nabla \alpha_s|^2 + |\Delta \alpha_s|^2 \\
& + \tau \big(|u|^2 + |\alpha|^2 + (1+\omega)|\nabla \alpha|^2 + |\nabla \alpha_s|^2 + |\nabla \theta|^2 + 2\nabla \alpha \nabla \alpha_s \big) \Big) dx \\
& + \int_0^t \int_{R(z)} \Big(|u_s|^2 + |\nabla \alpha_s|^2 + |\Delta \alpha_s|^2
\end{aligned}$$

$$+\tau\big(|\nabla u|^2+|\nabla\alpha|^2+|\alpha_{ss}|^2+|\nabla\alpha_{ss}|^2-|\nabla\alpha_s|^2\big)\Big)dx\,ds$$
$$+\frac{\omega}{2}\int_0^t\int_{R(z)}\Big(|\nabla u|^2+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2+|\alpha_s|^2+2|\nabla\alpha_s|^2$$
$$+|\Delta\alpha_s|^2+\tau\big(|u|^2+|\alpha|^2+(1+\omega)|\nabla\alpha|^2+|\nabla\alpha_s|^2+|\nabla\theta|^2\big)\Big)dx\,ds$$
$$+\tau\int_0^t\int_{R(z)}\Big(f(u)u+g(u)u_s\alpha_{ss}+G(u)\alpha-g(u)u(\alpha_s-\Delta\alpha_s)\Big)dx\,ds \tag{5.19}$$

and

$$E_\omega(z,t)=\frac{e^{-\omega t}}{2}\int_{R(z)}\Big(|\nabla u|^2+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2+|\alpha_s|^2+2|\nabla\alpha_s|^2+|\Delta\alpha_s|^2$$
$$+\tau\big(|u|^2+|\alpha|^2+(1+\omega)|\nabla\alpha|^2+|\nabla\alpha_s|^2+|\nabla\theta|^2+2\nabla\alpha\nabla\alpha_s\big)\Big)dx$$
$$+\int_0^t\int_{R(z)}e^{-\omega s}\Big(|u_s|^2+|\nabla\alpha_s|^2+|\Delta\alpha_s|^2$$
$$+\tau\big(|\nabla u|^2+|\nabla\alpha|^2+|\alpha_{ss}|^2+|\nabla\alpha_{ss}|^2-|\nabla\alpha_s|^2\big)\Big)dx\,ds$$
$$+\frac{\omega}{2}\int_0^t\int_{R(z)}e^{-\omega s}\Big(|\nabla u|^2+2F(u)+|\nabla\alpha|^2+|\Delta\alpha|^2+|\alpha_s|^2+2|\nabla\alpha_s|^2$$
$$+|\Delta\alpha_s|^2+\tau\big(|u|^2+|\alpha|^2+(1+\omega)|\nabla\alpha|^2+|\nabla\alpha_s|^2+|\nabla\theta|^2\big)\Big)dx\,ds$$
$$+\tau\int_0^t\int_{R(z)}e^{-\omega s}\Big(f(u)u+g(u)u_s\alpha_{ss}+G(u)\alpha-g(u)u(\alpha_s-\Delta\alpha_s)\Big)dx\,ds \tag{5.20}$$

with $R(z)=\{x\in R : x_1>z\}$. We get

Theorem 5.1. *Let* $(u,\alpha,\frac{\partial\alpha}{\partial t})$ *be a solution to problem* (1.4)-(1.7) *with the boundary conditions* (5.1)-(5.4). *Then, either this solution satisfies the asymptotic condition*

$$H_\omega(z,t)\geq e^{K^{-1}(z-z_0)}H_\omega(z_0,t),\quad z\geq z_0 \tag{5.21}$$

or it satisfies

$$\mathcal{E}_\omega(z,t)\leq E_\omega(0,t)e^{(\omega t-K^{-1}z)},\quad z\geq 0 \tag{5.22}$$

Proof. Using the estimate (5.18) and due to Phragmén-Lindelöf alternative we get, either there exists $z_0\geq 0$ such that $H_\omega(z_0,t)>0$ for all $z\geq z_0$ such that

$$H_\omega(z,t)\geq e^{K^{-1}(z-z_0)}H_\omega(z_0,t), z\geq z_0 \tag{5.23}$$

or $H_\omega(z_0,t)<0$ for all z that means

$$-H_\omega(z,t)\leq -e^{-K^{-1}z}H_\omega(0,t), z\geq 0. \tag{5.24}$$

The estimate (5.23) gives information in terms of the measure defined in the cylinder, this leads H_ω tends exponentially fast to infinity. Otherwise, the inequality (5.24) shows that $H_\omega(z,t)$ tends to zero, this implies

$$E_\omega(z,t) \leq E_\omega(0,t)e^{-K^{-1}z}, z \geq 0. \tag{5.25}$$

This achieves the proof. □

Remark 5.2. Estimates (5.21) and (5.22) are known respectively as growth and decay estimates.

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