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A NOTE ON BLOCK PRECONDITIONER FOR GENERALIZED SADDLE POINT MATRICES WITH HIGHLY SINGULAR (1,1) BLOCK

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Abstract In this paper, we present a block triangular preconditioner for generalized saddle point matrices whose coefficient matrices have singular (1,1) blocks. Theoretical analysis shows that all the eigenvalues of the preconditioned matrix are strongly clustered when choosing an optimal parameter. Numerical experiments are given to demonstrate the efficiency of the presented preconditioner.

Keywords Saddle point matrices, Krylov subspace methods, generalized saddle point matrices, minimal polynomial, preconditioners.

MSC(2010) 65F10, 65F15.

1. Introduction

Consider the following the generalized saddle point problem

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} A & B^T \\ C & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.1}$$

where $A \in \mathcal{R}^{n,n}, B, C \in \mathcal{R}^{m,n}, m \leq n$. The matrix $\mathcal{A}$ is assumed to be nonsingular, whereas the matrix A is singular with a high nullity. That is to say dim(kernel(A)) is larger. Systems of the form (1.1) arise in a variety of scientific and engineering applications and have attracted a lot of research, see [2–5, 28] for a comprehensive survey. We refer the reader to [1–9, 14–18, 24–30] for possible applications of generalized saddle point problems. We emphasize that while numerous effective solution algorithms exist for the case of a positive definite or semidefinite (1,1) block, relatively little has been done for the case where the (1,1) block is indefinite.

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Generally speaking, this is a rather challenging problem, which gets harder as the matrix A becomes more indefinite and makes the coefficient matrix $\mathcal{A}$ close to singular and therefore highly ill-conditioned.

In 2009, Huang, Cheng and Li [18] established two types of block triangular preconditioners applied to the linear saddle point problems with the singular (1,1) block. In 2006, for symmetric saddle point linear systems with (1,1) blocks that have a high nullity, Greif and Schötzau presented a Schur complement-free block diagonal preconditioner $\mathcal{M}_W$ based on augmentation in [15], and showed that the preconditioned matrix $\mathcal{M}_W^{-1}\mathcal{A}$ has only two distinct eigenvalues 1 and -1. Then, Cao [8] extended the preconditioner $\mathcal{M}_W$ of the generalized saddle point systems (1.1) to $\mathcal{D}_{Aug}$ and showed that $\mathcal{D}_{Aug}^{-1}\mathcal{A}$ still has two distinct eigenvalues 1 and -1. In 2007, for the symmetric saddle point case, Rees and Greif [26] introduced a more general augmentation block preconditioner $\mathcal{M}_k$. Recently, Cao [7] and Zhang [30] considered two augmentation block preconditioners $\mathcal{T}_{k,j}$ and $\mathcal{H}_{\xi,\eta}$. Moreover, they gave the results on the eigenvalue distribution, forms of the eigenvectors of the corresponding block preconditioned matrix and their minimal polynomial. Based on the preconditioners by Cao and Zhang [7, 30], we present a block triangular preconditioner, which is defined as follows:

$$\mathcal{P}_t = \begin{pmatrix} A + tB^TW^{-1}C & (1-t)B^T \\ 0 & W \end{pmatrix}, \tag{1.2}$$

where $W \in_R^{m,m}$ is symmetric positive definite and such that $A + tB^TW^{-1}C$ is invertible, t is a scalar. Generally speaking, the condition number $A + tB^TW^{-1}C$ is relatively large including special cases $t = -1$.

Remark 1.1. Obviously, the preconditioner $\mathcal{P}_t$ is different from the preconditioners $\mathcal{T}_{k,j}$ and $\mathcal{H}_{\xi,\eta}$. Note that the augmented Lagrangian formulation with the absolute value of t taken sufficiently large makes the (1, 1) block $A+tB^TW^{-1}C$ less asymmetric and indefinite; indeed, in the limit as $t \to \infty$ the symmetric positive semidefinite contribution $tB^TW^{-1}C$ will dominate the (1, 1) block. Moreover, a very large value of absolute value t is likely to make the block $A + tB^TW^{-1}C$ very ill-conditioned and therefore difficult to invert. Hence, the choice of the algorithmic parameter t involves a trade-off.

Based on the block triangular preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$, similar to the proving process of section 2 in [7, 30] we give the eigenvalue distribution with the presented preconditioner. Finally, numerical examples show that the block triangular preconditioner $\mathcal{P}_t$ has the same spectral clustering with preconditioners $\mathcal{T}_{k,j}$ and $\mathcal{H}_{\xi,\eta}$ when choosing the suitable parameters.

This paper is organized as follows. In Section 2, we will study the spectral analysis of the block triangular preconditioner for the saddle point system. One numerical example is given in Section 3. Finally, conclusions are made in Section 4.

2. Main results

Consider the following augmentation block preconditioner:

$$\mathcal{P}_t = \begin{pmatrix} A + tB^TW^{-1}C & (1-t)B^T \\ 0 & W \end{pmatrix},$$

where t is a scalar.

Now we consider the spectrum of the preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$.

Theorem 2.1. *Assume that $\mathcal{A}$ is nonsingular and its $(1,1)$ block A is singular with nullity $s(\leq m)$. Let $\{z_i\}_{i=1}^{n-m}$ be a basis of $\mathcal{N}(C)$, $\{x_i\}_{i=1}^{s}$ a basis of $\mathcal{N}(A)$. Then 1 is an eigenvalue of the preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$ of geometric multiplicity $n-m$, the corresponding eigenvectors are $\{[z_i^T, 0^T]^T\}_{i=1}^{n-m}$.*
(1) When $t \neq -1$, then $\lambda_1 = \frac{t-1+|t+1|}{2t}$ and $\lambda_2 = \frac{t-1-|t+1|}{2t}$ are two eigenvalues of $\mathcal{P}_t^{-1}\mathcal{A}$ both of geometric multiplicity s, the corresponding eigenvectors are

$$\left\{\left[x_i^T, (W^{-1}Cx_i)^T\right]^T \text{ and } \left[x_i^T, -t(W^{-1}Cx_i)^T\right]\right\}_{i=1}^{s},$$

respectively.
(2) When $t = -1$, then 1 is an eigenvalue of multiplicity $2s$, while the algebraic multiplicity is 2. 1 corresponds s eigenvectors $\{[x_i^T, (W^{-1}Cx_i)^T]\}_{i=1}^{s}$ and s generalized eigenvectors of order 2(Please refer to Chapter 13: Preconditioning [29]).

Proof. Let λ denote an eigenvalue of $\mathcal{P}_t^{-1}\mathcal{A}$ with eigenvector $[u^T, v^T]^T$. Hence

$$\begin{pmatrix} A & B^T \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} A + tB^TW^{-1}C & (1-t)B^T \\ 0 & W \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.1}$$

Expanding out (2.1) we obtain

$$\begin{cases} Au + B^Tv = \lambda(A + tB^TW^{-1}C)u + \lambda(1-t)B^Tv, \\ Cu = \lambda Wv. \end{cases} \tag{2.2}$$

Since $\mathcal{A}$ is nonsingular, it follows that $\lambda \neq 0$. Furthermore, we claim that $u \neq 0$. If not, from Eq. (2.2) we have $Wv = 0$. Since W is a symmetric positive definite matrix, then we can immediately get that $v = 0$. Hence, from the second equation of (2.2), we obtain

$$v = \frac{1}{\lambda}W^{-1}Cu. \tag{2.3}$$

Substituting (2.3) into the first equation of (2.2) yields

$$\lambda(1-\lambda)Au - \left[t\lambda^2 + (1-t)\lambda - 1\right] B^TW^{-1}Cu = 0. \tag{2.4}$$

Let $u \in \mathcal{N}(C)$, then Eq. (2.4) implies $\lambda(1-\lambda)Au = 0$. Since $\mathcal{N}(A) \cap \mathcal{N}(C) = \{0\}$ (please refer to Proposition 2.1 [8]), 1 is an eigenvalue of $\mathcal{P}_t^{-1}\mathcal{A}$ of geometric multiplicity $n-m$, the corresponding eigenvectors are $\{z_i^T, 0^T\}_{i=1}^{n-m}$.

Let $u \in \mathcal{N}(A)$, then Eq. (2.4) implies $\left[t\lambda^2 + (1-t)\lambda - 1\right] B^TW^{-1}Cu = 0$. Since $\mathcal{N}(A) \cap \mathcal{N}(C) = \{0\}$ (please refer to Proposition 2.1 [8]) and $\text{rank}(B) = m$, we obtain

$$t\lambda^2 + (1-t)\lambda - 1 = 0. \tag{2.5}$$

From Eq. (2.5) we have two roots

$$\lambda_1 = \frac{t-1+|t+1|}{2t} \text{ and } \lambda_2 = \frac{t-1-|t+1|}{2t}. \tag{2.6}$$

(1) If $t \neq -1$, we have: When $t > -1, \lambda_1 = 1, \lambda_2 = -\frac{1}{t}$; When $t < -1, \lambda_1 = -\frac{1}{t}, \lambda_2 = 1$. Then λ_1 and λ_2 are two distinct eigenvalues of $\mathcal{P}_t^{-1}\mathcal{A}$ both of geometric multiplicity s, the corresponding eigenvectors are

$$\left\{ \left[x_i^T, \frac{2t}{t-1+|t+1|}(W^{-1}Cx_i)^T \right]^T \text{ and } \left[x_i^T, \frac{2t}{t-1-|t+1|}(W^{-1}Cx_i)^T \right]^T \right\}_{i=1}^{s},$$

or

$$\left\{ \left[x_i^T, (W^{-1}Cx_i)^T \right]^T \text{ and } \left[x_i^T, -t(W^{-1}Cx_i)^T \right]^T \right\}_{i=1}^{s},$$

(2) If $t = -1$, then $\lambda_1 = \lambda_2 = \frac{t-1}{2t} = \frac{-2}{-2} = 1$ is an eigenvalue of multiplicity $2s$. While the geometric multiplicity is s. 1 corresponds s eigenvectors $\{[x_i^T, (W^{-1}Cx_i)^T]^T\}_{i=1}^{s}$ and s generalized eigenvectors of order 2. □

Remark 2.1. The preconditioner $\mathcal{P}_t$ in this paper and the preconditioners $\mathcal{T}_{k,j}, \mathcal{H}_{\xi,\eta}$ in [7,30] are three different preconditioning modes. Moreover, they have an intersection. That is to say, when choosing the appropriate parameters, the preconditioners $\mathcal{P}_t, \mathcal{T}_{k,j}, \mathcal{H}_{\xi,\eta}$ may become the same preconditioner.

Remark 2.2. From Theorem 2.1 we know that for any $t, \mathcal{P}_t^{-1}\mathcal{A}$ has eigenvalue 1 of multiplicity $n - m$. When $t = -1$, then $\mathcal{P}_t^{-1}\mathcal{A}$ has eigenvalue 1 of multiplicity $n - m + 2s$, very strong spectral clustering.

Corollary 2.2. *When $t = -1$, then preconditioner $\mathcal{P}_t$ is the optimal in the augmentation block preconditioner set $\{\mathcal{P}_t : t \text{ real}\}$.*

Corollary 2.3. *If $\operatorname{nullity}(A) = m$, then the minimal polynomial $p_t(\lambda)$ of the preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$ is*

$$p_t(\lambda) = \begin{cases} (\lambda - 1)^2(\lambda + \frac{1}{t}), & \textit{when } t \neq -1, \\ (\lambda - 1)^3, & \textit{when } t = -1. \end{cases} \tag{2.7}$$

3. Numerical examples

To further assess the effectiveness of the triangular block triangular preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$ combined with Krylov subspace methods, we present a sample of numerical example which is based on a two-dimensional time-harmonic Maxwell equation in mixed form in a square domain $(-1 \leq x \leq 1, -1 \leq y \leq 1)$. In all our runs we used as a zero initial guess and stopped the iteration when the relative residual had been reduced by at least six orders of magnitude (i.e, when $\|b - \mathcal{A}x^k\|_2 \leq 10^{-6}\|b\|_2$). For the simplicity, we take the generic source: $f = 1$ and a finite element subdivision such as Figure 1 based on uniform grids of triangle elements. Three mesh sizes are considered: $h = \frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{12}, \frac{\sqrt{2}}{18}$. The solutions of the preconditioned systems in each iteration are computed exactly. Information on the sparsity of relevant matrices on the different meshes is given in Table 1, where $\operatorname{nz}(A)$ denotes the nonzero elements of matrix A and m denotes the scale of the generalized saddle point problem.

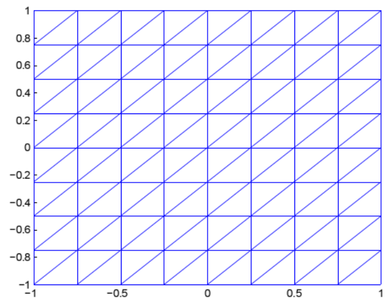

Figure 1. A uniform mesh with $h = \frac{\sqrt{2}}{4}$

Table 1. datasheet for different grids

Grid	m	n	nz(A)	nz(B)	nz(W)	order of $\mathcal{A}$
8×8	176	49	820	462	217	225
16×16	736	225	3556	2190	1065	961
32×32	3008	961	14788	9486	4681	3969
64×64	12160	3969	60292	39438	19593	16129

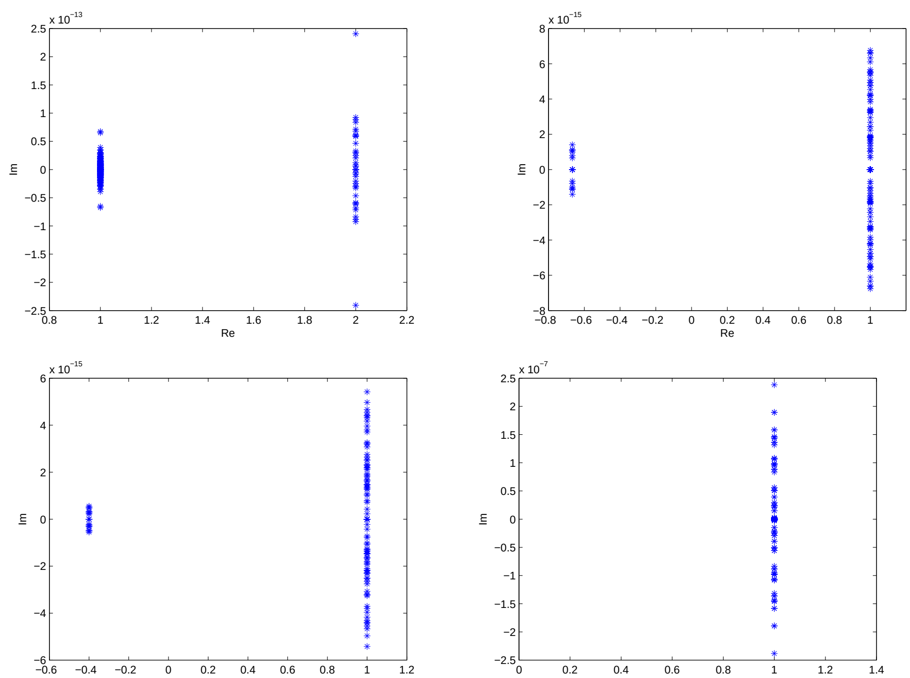

Figure 2. The eigenvalue distribution for the block triangular preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$ when $t = -0.5$(the first), $t = 1.5$(the second),$t = 2.5$(the third) and $t = -1$(the fourth,the optimal parameters), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

Since the new preconditioner has one parameter, in numerical experiments we will test different values. Numerical experiments show that the spectrum of the new preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$ is more strongly clustered when the parameters $t = -1$, which has the same eigenvalues aggregation as that of the preconditioned matrix $\mathcal{H}_{\xi,\eta}^{-1}\mathcal{A}$ [30] when $\xi\eta = -1$ and $\mathcal{T}_{k,j}^{-1}\mathcal{A}$ [14] when $k = 2, j = -1$.

In Figures 2, 3 and 4 we display the eigenvalues of the preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$ in the case of $h = \frac{\sqrt{2}}{8}$, $h = \frac{\sqrt{2}}{12}$ and $h = \frac{\sqrt{2}}{18}$ for different parameters. In Figures 5, 6, and 7 we display the eigenvalue distribution for the block triangular preconditioned matrix $\mathcal{T}_{k,j}^{-1}\mathcal{A}$, [7] $\mathcal{H}_{\xi,\eta}^{-1}\mathcal{A}$, [26] $\mathcal{P}_t^{-1}\mathcal{A}$ when choosing the optimal pamameters, respectively. Figures 2 $\sim$ 7 show that the distribution of eigenvalues of the preconditioned matrix confirms our above theoretical analysis.

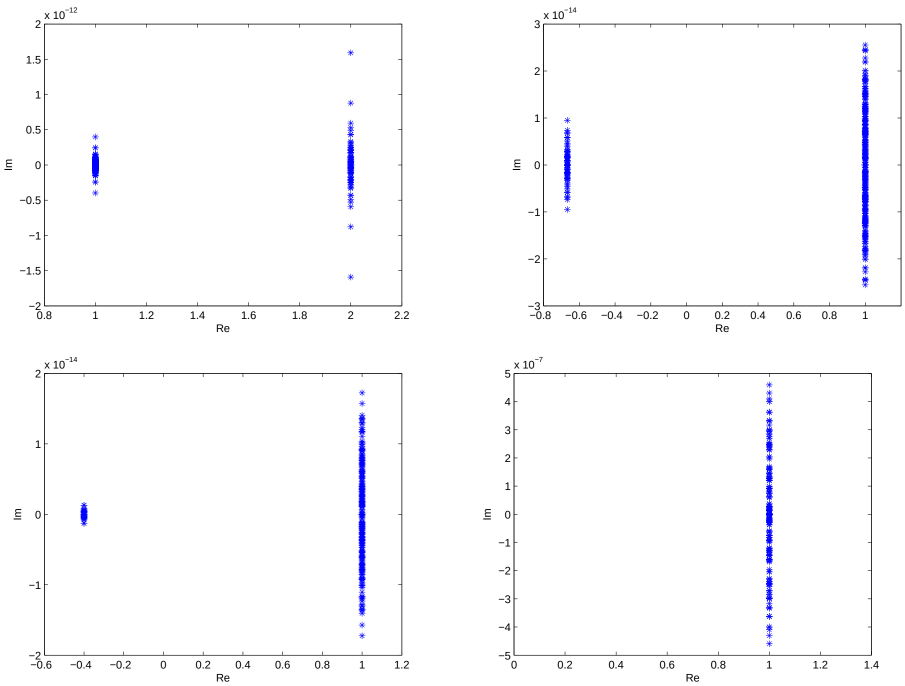

Figure 3. The eigenvalue distribution for the block triangular preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$ when $t = -0.5$(the first), $t = 1.5$(the second),$t = 2.5$(the third) and $t = -1$(the fourth,the optimal parameters), respectively. Here, $h = \frac{\sqrt{2}}{12}$.

In Tables 2 $\sim$ 4 we show iteration counts and relative residual about preconditioned matrices $\mathcal{P}_t^{-1}\mathcal{A}$, when choosing different parameters and applying to BICGSTAB and GMRES Krylov subspace iterative methods on three meshes, where $It_{BICGSTAB(\mathcal{P}_t^{-1}\mathcal{A})}$ and $Res_{BICGSTAB(\mathcal{P}_t^{-1}\mathcal{A})}$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{P}_t^{-1}\mathcal{A}$ when applying to BICGSTAB Krylov subspace iterative methods, respectively. $It_{GMRES(\mathcal{P}_t^{-1}\mathcal{A})}$ and $Res_{GMRES(\mathcal{P}_t^{-1}\mathcal{A})}$ are the iteration numbers and relative residual of the preconditioned matrices $\mathcal{P}_t^{-1}\mathcal{A}$ when applying to GMRES Krylov subspace iterative methods, respectively. $It_{BICGSTAB}$ and $Res_{BICGSTAB}$ are the iteration numbers and relative residual of unpreconditioned matrices when applying to BICGSTAB Krylov subspace iterative methods, respectively.

Remark 3.1. From the above figures and tables, we know that the block preconditioner $\mathcal{P}_t$ has the same spectral clustering as the preconditioner $\mathcal{H}_{\xi,\eta}$ [26] and $\mathcal{T}_{k,j}$

[7] when choosing the optimal parameters.

Table 2. Iteration counts and time about preconditioned matrices $\mathcal{P}_t^{-1}\mathcal{A}$ when choosing different parameters. Moreover, the contents of square brackets represent the iteration time (second). Here, $h = \frac{\sqrt{2}}{8}$ denotes the size of the corresponding grid.

t	$It_{BICGSTAB(\mathcal{P}_t^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_t^{-1}\mathcal{A})}$
-0.5	1.5[0.020]	3.2368×10^{-15}
1.5	1.5[0.018]	9.8030×10^{-15}
2.5	1.5[0.019]	1.3409×10^{-14}
-1	1[0.013]	2.2374×10^{-8}
t	$It_{GMRES(\mathcal{P}_t^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_t^{-1}\mathcal{A})}$
-0.5	2(1)[0.032]	8.4309×10^{-14}
1.5	2(1)[0.032]	8.1545×10^{-15}
2.5	2(1)[0.029]	6.9073×10^{-15}
-1	2(1)[0.028]	2.4741×10^{-14}

Table 3. Iteration counts and time about preconditioned matrices $\mathcal{P}_t^{-1}\mathcal{A}$ when choosing different parameters. Moreover, the contents of square brackets represent the iteration time (second). Here, $h = \frac{\sqrt{2}}{12}$ denotes the size of the corresponding grid.

t	$It_{BICGSTAB(\mathcal{P}_t^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_t^{-1}\mathcal{A})}$
-0.5	1.5[0.143]	8.6413×10^{-15}
1.5	1.5[0.155]	2.3759×10^{-14}
2.5	1.5[0.156]	3.3164×10^{-14}
-1	1[0.108]	2.0745×10^{-9}
t	$It_{GMRES(\mathcal{P}_t^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_t^{-1}\mathcal{A})}$
-0.5	2(1)[0.231]	2.8502×10^{-13}
1.5	2(1)[0.219]	9.1310×10^{-14}
2.5	2(1)[0.217]	2.7118×10^{-14}
-1	2(1)[0.213]	3.7082×10^{-13}

Table 4. Iteration counts and time about preconditioned matrices $\mathcal{P}_t^{-1}\mathcal{A}$ when choosing different parameters. Moreover, the contents of square brackets represent the iteration time (second). Here, $h = \frac{\sqrt{2}}{18}$ denotes the size of the corresponding grid.

t	$It_{BICGSTAB(\mathcal{P}_t^{-1}\mathcal{A})}$	$Res_{BICGSTAB(\mathcal{P}_t^{-1}\mathcal{A})}$
-0.5	1.5[1.5810]	2.6061×10^{-14}
1.5	1.5[1.567]	4.7794×10^{-14}
2.5	1.5[1.560]	$7/7002 \times 10^{-14}$
-1	1[1.040]	1.8904×10^{-10}
t	$It_{GMRES(\mathcal{P}_t^{-1}\mathcal{A})}$	$Res_{GMRES(\mathcal{P}_t^{-1}\mathcal{A})}$
-0.5	2(1)[2.105]	1.6401×10^{-13}
1.5	2(1)[2.149]	1.0922×10^{-13}
2.5	2(1)[2.056]	1.0281×10^{-13}
-1	2(1)[2.051]	4.3227×10^{-13}

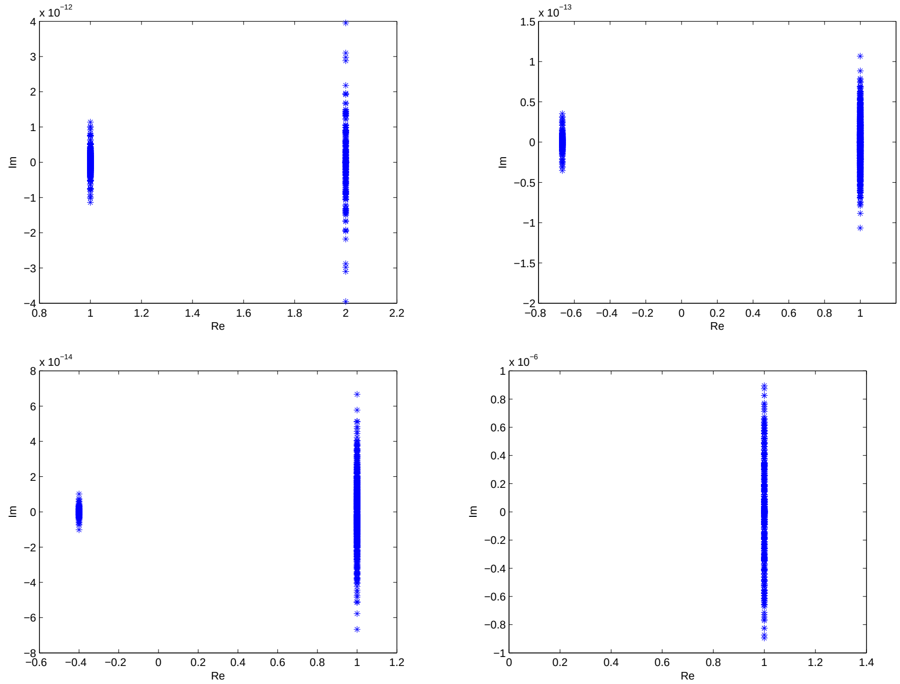

Figure 4. The eigenvalue distribution for the block triangular preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$ when $t = -0.5$(the first), $t = 1.5$(the second),$t = 2.5$(the third) and $t = -1$(the fourth,the optimal parameters), respectively. Here, $h = \frac{\sqrt{2}}{18}$.

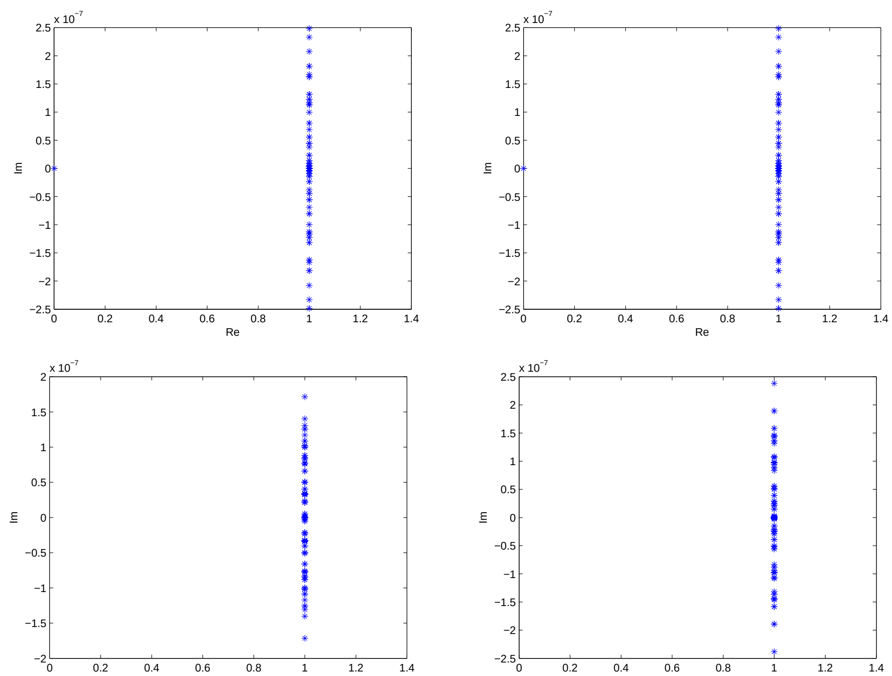

Figure 5. The eigenvalue distribution for the block triangular preconditioned matrix $\mathcal{T}_{k,j}^{-1}\mathcal{A}$ when $k = 2, j = -1$(the first,the optimal parameters), $\mathcal{H}_{\xi,\eta}^{-1}\mathcal{A}$ when $\xi = 1, \eta = -1$(the second, the optimal parameters $(\xi\eta = -1)$) and $\xi = -2, \eta = 1/2$(the third, the optimal parameters $(\xi\eta = -1)$), $\mathcal{P}_t^{-1}\mathcal{A}$ when $t = -1$(the fourth, the optimal parameters), respectively. Here, $h = \frac{\sqrt{2}}{8}$.

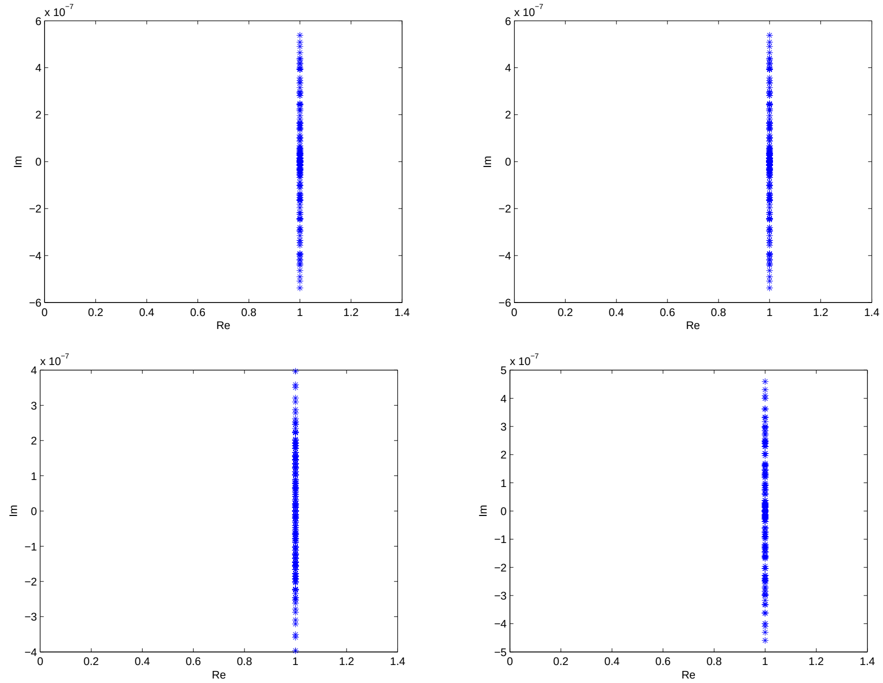

Figure 6. The eigenvalue distribution for the block triangular preconditioned matrix $\mathcal{T}_{k,j}^{-1}\mathcal{A}$ when $k=2, j=-1$(the first,the optimal parameters), $\mathcal{H}_{\xi,\eta}^{-1}\mathcal{A}$ when $\xi=1, \eta=-1$(the second, the optimal parameters $(\xi\eta=-1)$) and $\xi=-2, \eta=1/2$(the third, the optimal parameters $(\xi\eta=-1)$), $\mathcal{P}_t^{-1}\mathcal{A}$ when $t=-1$(the fourth, the optimal parameters), respectively. Here, $h=\frac{\sqrt{2}}{12}$.

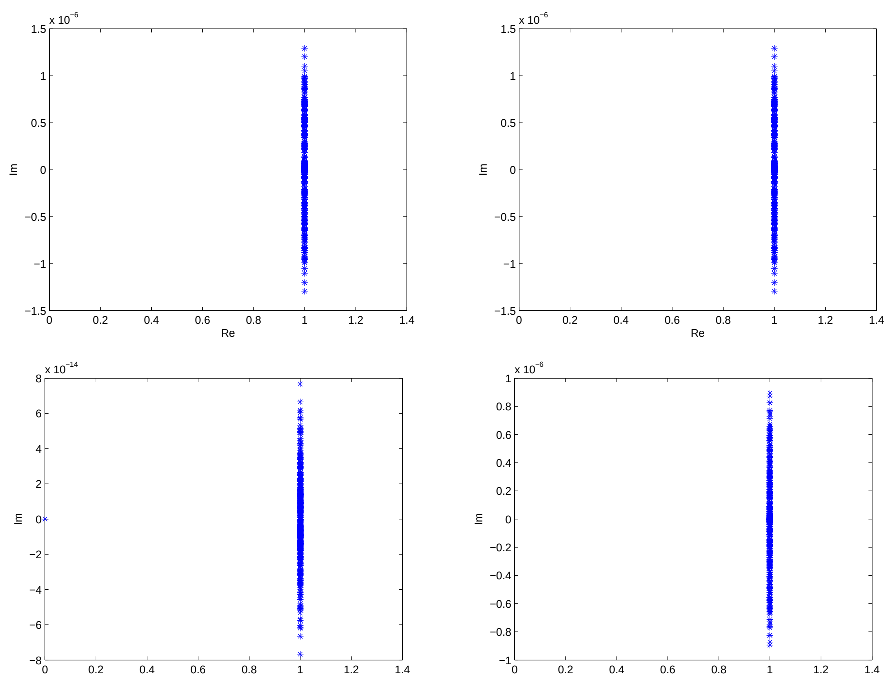

Figure 7. The eigenvalue distribution for the block triangular preconditioned matrix $\mathcal{T}_{k,j}^{-1}\mathcal{A}$ when $k=2, j=-1$(the first,the optimal parameters), $\mathcal{H}_{\xi,\eta}^{-1}\mathcal{A}$ when $\xi=1, \eta=-1$(the second, the optimal parameters $(\xi\eta=-1)$) and $\xi=-2, \eta=1/2$(the third, the optimal parameters $(\xi\eta=-1)$), $\mathcal{P}_t^{-1}\mathcal{A}$ when $t=-1$(the fourth, the optimal parameters), respectively. Here, $h=\frac{\sqrt{2}}{18}$.

4. Conclusions

In this paper, based on the preconditioners presented by Cao and Zhang [7, 30], we present a block preconditioner for generalized saddle point matrices whose coefficient matrices have singular (1,1) blocks. Moreover, theoretical analysis and numerical examples show that the eigenvalues of the preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$ is strongly clustered when $t = -1$.

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