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STABILITY ANALYSIS OF TIME DELAYED FRACTIONAL ORDER PREDATOR-PREY SYSTEM WITH CROWLEY-MARTIN FUNCTIONAL RESPONSE*

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Abstract In this paper, we present a fractional order predator-prey system with Crowley-Martin functional response. Firstly, we analyze the asymptotic stability of the system. At the same time, some sufficient conditions for the stability of the system are given. Then, we investigate the stability of the corresponding system with time delay and also discuss some sufficient conditions for the equilibrium stability of the system with time delay. In the end, the numerical simulations illustrate the accuracy of our conclusions.

Keywords Fractional order system, predator-prey model, time delay, Crowley-Martin functional response, stability analysis.

MSC(2010) 34D20, 92D25.

1. Introduction

Population dynamics is one of the main contents of biological mathematics and theoretical ecology. The research of population dynamics originates from demography, fishery resource and the application of zoology. In order to predict the trend of population, researchers use dynamic model to describe the change of population. Because of the establishment of modern differential equation theory, researchers have put forward various kinds of differential equation models. Since then, mathematicians and ecologists have studied various predator-prey models. So far, there have been a great deal of research results on the predator-prey model [3]. A general predator-prey model is given by

$$\begin{cases} \frac{dN_1}{dt} = N_1 f(N_1) - h(N_1, N_2)N_2, \\ \frac{dN_2}{dt} = ch(N_1, N_2)N_2 - mN_2, \end{cases} \tag{1.1}$$

where N_1 and N_2 represent the population density of the prey and predator, respectively; c and m indicate the conversion rate and death rate, respectively; the function $f(N_1)$ is the growth pattern of the prey population without predators; $h(N_1, N_2)$ is the functional response of predator and denotes the average feeding

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rate of a predator. The classical predator-prey model is given by Lotka and Volterra (L-V model)

$$\begin{cases} \frac{dN_1}{dt} = AN_1 - BN_1N_2, \\ \frac{dN_2}{dt} = BN_1N_2 - DN_2, \end{cases} \tag{1.2}$$

where A and D represent the birth rate of prey and the death rate of predator, respectively. The L-V model assumes that the average predation rate of each predator is a linear growth model. With the increase of the prey, the average predation ability of per predator continues to increase. This is not in line with the actual situation. In view of this situation, some researches have proposed many different types of functional response: Holling type [8], Holling type II [17,18,25] and Holling type III [6,7]. According to the above literature, we can find that Holling type I-III functional response is more reasonable than L-V functional response. However, there is not the effect of predator among Holling types I-III. Actually, with the increase of predator, there will occur competition among predators. That is to say, functional response will also be affected by the population density of predator. Therefore, some researchers have discussed functional response with predator dependence (Beddington-DeAngelis [2, 19, 21] and Crowley-Martin [14]). In the Beddinton-DeAngelis functional response and Crowley-Martin functional response, we can find that the average predation rate of per predator will be affected by the density of predator. Skalski and Gilliam [15] presented statistical evidence from 19 predator-prey systems and proved that three predator-dependent functional response (Beddington-DeAngelis, Crowley-Martin and Hassell-Varley) could provide better description of predator feeding over a range of predator-prey abundances. In some cases, the Crowley-Martin type preformed better among them, which is given blow

$$h(N_1, N_2) = \frac{p_1N_1}{(1 + p_2N_1)(1 + p_3N_2)}, \tag{1.3}$$

where p_1, p_2 and p_3 are positive parameters and denote effects of capture rate, handling time and mutual interference among predators, respectively. Therefore, in this paper, we consider Crowley-Martin functional response in the system.

In recent years, the fractional order systems have attracted the attention of many scholars. The fractional calculus has been proved to be an effective tool for system modeling in physical, biological, economic and other fields. Fractional differential equation is very suitable for describing materials and processes with memory and heredity. It has the advantages of simple modeling, clear physical meaning of parameters and accurate description for complex systems. Because of the heredity and memory of the fractional order system, fractional order population system can better reflect the historical process of population development. Ahmed researched the fractional order Lotka-Volterra predator-prey model and obtained the conditions for the existence and uniqueness of the solution [1]. Rihan discussed the stability of fractional order time delayed predator-prey systems with Holling type II functional response [13]. Chinnathambi considered the stability of fractional order prey-predator system with time delay and Monod-Haldane functional response [4]. Based on the previous analysis, we know that the fractional order predator-prey system with Crowley-Martin functional response is important, which can better response the changes of the population in some cases. However, there are few results on fractional order predator-prey system with Crowley-Martin functional response. Therefore, we discuss fractional order predator-prey system with Crowley-Martin functional response in this paper.

The structure of this paper is as follows. Section 2 presents the definition of fractional order derivative. Section 3 gives the stability analysis of the fractional order predator-prey system with Crowley-Martin functional response. In Section 4, we investigate the time delayed fractional-order predator-prey system with Crowley-Martin response function, and give sufficient conditions for stability of the system. In Section 5, numerical simulations illustrate the theoretical results.

2. Preliminaries

There are three kinds of fractional derivative commonly used: Riemann-Liouville derivative, Grunwald-Letnikov derivative and Caputo fractional derivative. The Caputo fractional-order derivative is adopted in this paper.

Definition 2.1 ([12])**.** The Caputo derivative of fractional order q for a function $f(t)$ is described as

$$ {}_0D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau, \tag{2.1}$$

where $\Gamma(\cdot)$ denotes the Gamma function and n is a positive integer such that $n-1 < q \le n$.

Proposition 2.1 ([12])**.** *The Laplace transform of the Caputo fractional order derivative is given by*

$$\mathcal{L}\{{}_0D_t^q f(t); s\} = s^q F(s) - \sum_{k=0}^{n-1} s^{q-k-1} f^{(k)}(t_0), \qquad n-1 < q \le n,$$

where $\mathcal{L}\{\cdot\}$ denotes the Laplace transform.

Consider the following fractional order time delayed linear system

$$ {}_0D_t^q X(t) = AX(t) + HX(t-\tau), \tag{2.2}$$

where $X(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ is a state vector; $X(t-\tau) = (x_1(t-\tau), x_2(t-\tau), \cdots, x_{n-\tau}(t))$ denotes a state vector with time delay; $A = (a_{ij})_{n\times n}$ represents the coefficient matrix; $H = (h_{i,j})_{n*n}$ indicates the coefficient matrix of the time delay term; $\tau > 0$ is the time delay.

Using Laplace transform on both sides of the system (2.2), we can obtain

$$\begin{cases} s^q Y_1(s) - s^{q-1}\phi_1(0) = h_{11}e^{-s\tau}(Y_1(s) + \int_{-\tau}^0 e^{-st}\phi_1(t)dt) \\ + \cdots + h_{1n}e^{-s\tau}(Y_n(s) + \int_{-\tau}^0 e^{-st}\phi_n(t)dt) + a_{1n}Y_n(s) + a_{11}Y_1(s), \\ s^q Y_2(s) - s^{q-1}\phi_2(0) = h_{21}e^{-s\tau}(Y_1(s) + \int_{-\tau}^0 e^{-st}\phi_1(t)dt) \\ + \cdots + h_{2n}e^{-s\tau}(Y_n(s) + \int_{-\tau}^0 e^{-st}\phi_n(t)dt) + a_{2n}Y_n(s) + a_{22}Y_2(s), \\ \qquad \vdots \\ s^q Y_n(s) - s^{q-1}\phi_n(0) = h_{n1}e^{-s\tau}(Y_1(s) + \int_{-\tau}^0 e^{-st}\phi_1(t)dt) + a_{n1}Y_1(s) \\ + \cdots + h_{nn}e^{-s\tau}(Y_n(s) + \int_{-\tau}^0 e^{-st}\phi_n(t)dt) + a_{nn}Y_n(s), \end{cases} \tag{2.3}$$

where $Y_i(s) = \mathcal{L}(x_i(t))$ $(i = 1, 2, \cdots, n)$ and $\phi_i(t)$ $(t \in [-\tau, 0])$ is initial condition. We write formula (2.3) in the form of vectors as follows

$$\triangle(s) \cdot Y(s) = d(s),$$

where $Y(s) = (Y_1(s), Y_2(s), \cdots, Y_n(s))^T$ is the Laplace transform of the state vector $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^t$ and $d(s) = (d_1(s), d_2(s), \cdots, d_n(s))$ denotes the nonlinear item remaining in system (2.3).
By calculating, it can be obtained that

$$\triangle(s) = \begin{bmatrix} s^q - h_{11}e^{-s\tau} - a_{11} & -h_{12}e^{-s\tau} - a_{12} & \cdots & -h_{1n}e^{-s\tau} - a_{1n} \\ -h_{21}e^{-s\tau} - a_{21} & s^q - h_{22}e^{-s\tau} - a_{22} & \cdots & -h_{2n}e^{-s\tau} - a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -h_{n1}e^{-s\tau} - a_{n1} & -h_{n2}e^{-s\tau} - a_{n2} & \cdots & s^q - h_{nn}e^{-s\tau} - a_{nn} \end{bmatrix}.$$

If time delay $\tau = 0$, system (2.2) becomes

$${}_0D_t^q X(t) = AX(t) + HX(t) = MX(t),$$

where

$$M = \begin{bmatrix} h_{11} + a_{11} & h_{12} + a_{12} & \cdots & h_{1n} + a_{1n} \\ h_{21} + a_{21} & h_{22} + a_{22} & \cdots & h_{2n} + a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} + a_{n1} & h_{n2} + a_{n2} & \cdots & h_{nn} + a_{nn} \end{bmatrix}.$$

Lemma 2.1 ([22]). *(Descartes rule of signs): Let $p(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ denote a polynomial with nonzero real coefficients a_i, where the b_i are integers satisfying $0 \le b_0 < b_1 < b_2 < \cdots < b_n$. Then the number of positive real zero of $p(x)$ (counted with multiplicities) is either equal to the number of variations in sign in the sequence $a_0, \ldots, a_n$ of the coefficients or less than that by an even whole number. The number of negative zeros of $p(x)$ (counted with multiplicities) is either equal to the number of variations in sign in the sequence of the coefficients of $p(-x)$ or less than that by an even whole number.*

Lemma 2.2 ([10]). *(Routh-Hurwitz Criterion): All roots of the polynomial $a_ns^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0$ have strictly negative real parts if and only if $a_i > 0$ for $i = 0, 1, 2, \cdots, n$ and the first column elements of the Routh-Hurwitz table are positive.*

For system (2.2), we have the following lemmas.

Lemma 2.3 ([5]). *If $q \in (0, 1)$ and all the roots of the characteristic equation $det(\triangle(s)) = 0$ have negative real part, then the zero solution of the system (2.2) is Lyapunov globally asymptotically stable.*

Lemma 2.4 ([9]). *If $q \in (0, 1)$, $x(t) \in R^n$ and $M \in R^{n\times n}$. Consider the following linear fractional-order system*

$${}_0D_t^q X(t) = MX(t),$$

Suppose $\lambda_i (i = 1, 2, \cdots, n)$ are all eigenvalues of the matrix $|\lambda I - M| = 0$. The linear fractional-order system is asymptotically stable if and only if $|\lambda_i| > \frac{q\pi}{2}$, and is stable if and only if $|\lambda_i| \geqslant \frac{q\pi}{2}$.

The stability region of a linear fractional-order system with order q ($q \in (0, 1)$) are depicted in Figure 1.

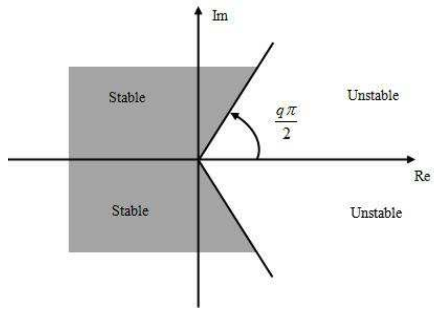

Figure 1. The stability region of a linear fractional-order system with order q ($q \in (0, 1)$).

Lemma 2.5 ([23]). *If $q \in (0, 1)$, all the eigenvalus of M satisfy $|arg(\lambda)| > \frac{\pi}{2}$ and the characteristic equation $\triangle(s) = 0$ has not pure imaginary roots for $\tau > 0$, then the zero solution of system (2.2) is Lyapunov asymptotically stable.*

Remark 2.1. For system (2.2), if all eigenvalues of the coefficient matrix M satisfy $|arg(\lambda)| > \frac{\pi}{2}$, and characteristic equation $det(\triangle(s)) = 0$ for any $\tau > 0$ has not pure imaginary roots. Then, the zero solution of the system (2.2) is stable. Noticing Lemma 2.4, if all eigenvalues of the coefficient matrix M satisfy $|arg(\lambda)| > \frac{q\pi}{2}$. The zero solution of the system (2.2) is not necessarily stable. The corresponding counter examples can be cited. Details are discussed in [24]. Therefore, the stability of time delay systems is judged by condition $|arg(\lambda)| > \frac{\pi}{2}$ instead of condition $|\lambda_i| > \frac{q\pi}{2}$.

3. Local asymptotic stability of the system

Through the above analysis, we consider the following predator-prey model

$$\begin{cases} {}_0D_t^q N_1(t) = N_1(A - BN_1 - K_1(N1, N2)), \\ {}_0D_t^q N_2(t) = N_2(-D - EN_2 + K_2(N1, N2)), \end{cases} \tag{3.1}$$

where $q \in (0, 1)$; N_1 and N_2 denote the density of prey and predator at time t, respectively; A and B denote the birth rate and intraspecific competition rate of prey, respectively; C is the predation rate; D and E represent the death rate and intraspecific competition rate of predator, respectively; F represents conversion rate of prey; the initial conditions $N_1(0) = N_{10} > 0, N_2(0) = N_{20} > 0$; $K_1(N1, N2) = \frac{CN_2}{A_1 + B_1N_1 + C_1N_2 + B_1C_1N_1N_2}$; $K_2(N1, N2) = \frac{FN_1}{A_1 + B_1N_1 + C_1N_2 + B_1C_1N_1N_2}$. All the parameters $A, B, C, D, E, F, A_1, B_1, C_1$ are assumed positive. For the convenience of the later discussion, transform the system (3.1) into the following system

$$\begin{cases} {}_0D_t^q x(t) = x(1 - x - k_1(x, y)), \\ {}_0D_t^q y(t) = y(-d - ey + k_2(x, y)), \end{cases} \tag{3.2}$$

where $k_1(x,y) = \frac{cy}{1+a_1x+b_1y+c_1xy}$, $k_2(x,y) = \frac{fx}{1+a_1x+b_1y+c_1xy}$, $N_1 = \frac{Ax}{B}$, $y = N_2$, $t = AT$, $x(0) = x_0 > 0, y(0) = y_0 > 0$, $c = \frac{C}{AA_1}$, $a_1 = \frac{AB_1}{A_1B}$, $b_1 = \frac{C_1}{A_1}$, $c_1 = \frac{AB_1C_1}{A_1B}$, $d = \frac{D}{A}$, $e = \frac{E}{A}$ and $f = \frac{FA}{AA_1B}$.

$E_1 = (0,0)$ and $E_2 = (1,0)$ are obviously the two nonnegative equilibrium points of the system (3.2). For the nontrivial equilibrium point $E_3 = (x^*, y^*)$, we have the following analysis.

Theorem 3.1. *The component x^* in the nontrivial equilibrium point E_3 of system (3.2) is positive, if $b_1 < c$.*

Proof. The interior equilibrium E_3 of system (3.2) is given by

$$1 - x - \frac{cy}{1 + a_1x + b_1y + c_1xy} = 0, \tag{3.3}$$

$$-d - ey + \frac{fx}{1 + a_1x + b_1y + c_1xy} = 0. \tag{3.4}$$

It can be obtained by (3.3)

$$y = \frac{(1 + a_1x)(1 - x)}{c - b_1 + b_1x - c_1x + c_1x^2}. \tag{3.5}$$

Taking (3.5) into (3.4), we have

$$\upsilon_1x^5 + \upsilon_2x^4 + \upsilon_3x^3 + \upsilon_4x^2 + \upsilon_5x + \upsilon_6 = 0, \tag{3.6}$$

where
$\upsilon_1 = c_1^2f$, $\upsilon_2 = 2c_1f(b_1 - c)$, $\upsilon_3 = a_1c(a_1e - c_1d) + f(b_1^2 + c_1^2 + 2cc_1 - 4b_1c_1)$, $\upsilon_4 = c(-c_1d + a_1d(c_1 - b_1)) - a_1ce(a_1 - 2) + f(2b_1c - 2cc_1 - 2b_1^2 + 2b_1c_1)$, $\upsilon_5 = cd(c_1 - b_1) + a_1cd(b_1 - c) + ce - 2a_1ce + f(c^2 + b_1^2 - 2b_1c)$ and $\upsilon_6 = cd(b_1 - c) - ce$. x^* in equilibrium point $E_3 = (x^*, y^*)$ is a solution of equation (3.6). According to Lemma 2.1 (Descartes' rule of signs) and assuming $b_1 < c$, we have $\upsilon_2 < 0$ and $\upsilon_6 < 0$. And we can find the symbols of υ_1, υ_2, υ_3, υ_4, υ_5, υ_6 change odd times, there must be a positive root x^* in the equation (3.6). □

According to the proof of Theorem 3.1, we can get the corresponding $y^* = \frac{(1+a_1x^*)(1-x^*)}{c-(b_1+c_1x^*)(1-x^*)}$. If conditions $b_1 < c$, $c > (b_1 + c_1x^*)(1 - x^*)$ and $0 < x^* < 1$ are satisfied, we can get $y^* > 0$. Therefore, the equilibrium point E_3 of the system (3.2) is the positive equilibrium point.

Theorem 3.2. *With respect to system (3.2), we have that*

(i) the equilibrium point $E_1 = (0,0)$ is unstable;

(ii) the equilibrium point $E_2 = (1,0)$ is locally asymptotically stable, if $d > \frac{f}{1+a_1}$;

(iii) the equilibrium point $E_3 = (x^*, y^*)$ is locally asymptotically stable, if $a_1(b_1 - c) > c_1$.

Proof. Assuming that the equilibrium point of (3.2) is $(\bar{x}, \bar{y})$ and letting $Z(t) = (x(t), y(t))$, we can obtain the linearize of the system (3.4)

$${}_0D_t^qZ(t) = AZ(t) + BZ(t), \tag{3.7}$$

with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -d \end{bmatrix}, B = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix},$$

where

$$l_{11} = -2\bar{x} + \frac{c(a_1 + c_1\bar{y})\bar{x}\bar{y} - c\bar{y}k_3(\bar{x},\bar{y})}{k_4(\bar{x},\bar{y})},\ l_{12} = \frac{c\bar{x}\bar{y}(b_1 + c_1\bar{x}) - c\bar{x}k_3(\bar{x},\bar{y})}{k_4(\bar{x},\bar{y})},$$

$$l_{21} = \frac{f\bar{y}k_3(\bar{x},\bar{y}) - (a_1 + c_1\bar{y})f\bar{x}\bar{y}}{k_4(\bar{x},\bar{y})} \text{ and } l_{22} = -2e\bar{y} + \frac{f\bar{x}k_3(\bar{x},\bar{y}) - f\bar{x}\bar{y}(b_1 + c_1\bar{x})}{k_4(\bar{x},\bar{y})},$$

with $k_3(\bar{x},\bar{y}) = 1 + a_1\bar{x} + b_1\bar{y} + c_1\bar{x}\bar{y}$ and $k_4(\bar{x},\bar{y}) = (1 + a_1\bar{x} + b_1\bar{y} + c_1\bar{x}\bar{y})^2$.

We can obtain the corresponding characteristic matrix

$$\triangle(s) = \begin{bmatrix} s^q - 1 - l_{11} & -l_{12} \\ -l_{21} & s^q + d - l_{22} \end{bmatrix},$$

and characteristic equation

$$det(\triangle(s)) = (s^q - 1 - l_{11})(s^q + d - l_{22}) - l_{12}l_{21} = 0. \tag{3.8}$$

Taking $E_1 = (0, 0)$ into (3.8), we can obtain

$$det(\triangle(s)) = (s^q - 1)(s^q + d) = 0.$$

It can be obtained that $s^q - 1 = 0$ or $s^q + d = 0$. The characteristic equation has a positive root $s^q = 1$. According to Lemma 2.4, the equilibrium point E_1 is unstable. Similarly, with respect to E_2, we have

$$det(\triangle(s)) = (s^q + 1)(s^q + d - \frac{f}{1 + a_1}) = 0. \tag{3.9}$$

Letting $s^q = g$, we have

$$det(\triangle(s)) = (g + 1)(g + d - \frac{f}{1 + a_1}) = 0. \tag{3.10}$$

It is clear that $Re(s) < 0$ in (3.9) is equivalent to $|arg(g)| > \frac{q\pi}{2}$ in (3.10)(see Ref. [16]). The characteristic equation $(g + 1)(g + d - \frac{f}{1+a_1}) = 0$ has a clear root $g_1 = -1$, and g_1 satisfies $|arg(g_1)| > \frac{q*\pi}{2}(q \in (0, 1))$. Assuming $d - \frac{f}{1+a_1} > 0$, anothor root $g_2 < 0$ satisfies $|arg(g)| > \frac{q*\pi}{2}(q \in (0, 1))$. According to Lemma 2.3 and Lemma 2.4, if $d - \frac{f}{1+a_1} > 0$, the equilibrium point $E_2 = (1, 0)$ is locally asymptotically stable.

For E_3, taking $x(t) = x^* + z(t)$ and $y(t) = y^* + w(t)$ into (3.2), we have

$$\begin{cases} {}_0D_t^q z(t) = [x^* + z(t)] - [x^* + z(t)]^2 \frac{c[x^*+z(t)][y^*+w(t)]}{k_5(z(t),w(t))}, \\ {}_0D_t^q w(t) = -d[y^* + w(t)] - e[y^* + w(t)]^2 + \frac{f[x^*+z(t)][y^*+w(t)]}{k_5(z(t),w(t))}, \end{cases} \tag{3.11}$$

where $k_5(z(t), w(t)) = 1+a_1[x^*+z(t)]+b_1[y^*+w(t)]+c_1[x^*+z(t)][y^*+w(t)]$. We can get the linearization matrix of system (3.11) at the equilibrium point $E_3 = (x^*, y^*)$

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix},$$

where $h_{11} = \frac{(a_1+c_1y^*)cx^*y^*}{k_6(x,y)} - x^*$, $h_{12} = \frac{-cx^*(1+a_1x^*)}{k_6(x,y)}$, $h_{21} = \frac{fy^*(1+b_1y^*)}{k_6(x,y)}$ and $h_{22} = -\frac{fx^*y^*(b_1+c_1x^*)}{k_6(x,y)} - ey^*$, with $k_6(x,y) = (1 + a_1x^* + b_1y^* + c_1x^*y^*)^2$. It is obvious that $h_{12} < 0$, $h_{21} > 0$ and $h_{22} < 0$. Assuming $h_{11} < 0$, we can obtain

$$\begin{cases} s_1 + s_2 = tr(H) = h_{11} + h_{22} < 0, \\ s_1 s_2 = det(H) = h_{11}h_{22} - h_{12}h_{21} > 0, \end{cases} \tag{3.12}$$

where s_1 and s_2 are the roots of the characteristic equation $|\lambda E - H| = 0$. Therefore, we can get $Re(s_1) < 0$ and $Re(s_2) < 0$. According to Lemma 2.3, E_3 is locally asymptotically stable. Based on the above analysis, we need to find the conditions to meet $h_{11} < 0$.

Assuming $h_{11} < 0$, we can obtain

$$-1 + \frac{cy^*(a_1 + c_1y^*)}{(1 + a_1x^* + b_1y^* + c_1x^*y^*)^2} < 0. \tag{3.13}$$

Furthermore, we have the equilibrium equation

$$1 + a_1x^* + b_1y^* + c_1x^*y^* = \frac{cy^*}{1 - x^*}. \tag{3.14}$$

Then, we have

$$y^* = \frac{(1 + a_1x^*)(1 - x^*)}{c - (1 - x^*)(b_1 + c_1x^*)}. \tag{3.15}$$

Taking (3.14) into (3.13), we can obtain

$$\frac{(1 - x^*)^2(a_1 + c_1y^*)}{cy^*} < 1. \tag{3.16}$$

Taking (3.15) into (3.16), we can get

$$\frac{(1 - x^*)^2(a_1 + c_1y^*)[c - (1 - x^*)(b_1 + c_1x^*)]}{c(1 + a_1x^*)(1 - x^*)} < 1. \tag{3.17}$$

If $0 < x^* < 1$, it can be obtained that

$$a_1[c - (1 - x^*)(b_1 + c_1x^*)] < \frac{c(1 + a_1x^*)}{1 - x^*} - c_1(1 + a_1x^*)(1 - x^*). \tag{3.18}$$

We assume

$$(1 + a_1x^*)[c - c_1(1 - x^*)] > a_1[c - (1 - x^*)(b_1 + c_1x^*)]. \tag{3.19}$$

Under (3.19), we can find the inequality (3.18) is satisfied. Furthermore, inequality (3.19) is equivalent to

$$[c_1 + a_1(c - b_1)](x^* - 1) + c > 0.$$

Therefore, the condition $c_1 + a_1(c - b_1) < 0$ can guarantee the establishment of $h_{11} < 0$. Therefore, the equilibrium point E_3 is locally asymptotically stable. If $x^* > 1$, according to (3.15), we have

$$y^* = \frac{(1 + a_1x^*)(1 - x^*)}{c - (1 - x^*)(b_1 + c_1x^*)} < 0. \tag{3.20}$$

Therefore, the case is not considered.

If $x^* = 1$, according to (3.16), we have $y^* = 0$. Then, the equilibrium point E_3 and the equilibrium point E_2 are the same point. $\square$

4. Local asymptotic stability of the time delayed system

In the early research of predator-prey model, it is assumed that the predation process of predator increased its population density instantaneously. However, the change of density of population is effected by not only the current activity, but also the past state of the population. We assume that the reproduction of predator after predating the prey will not be instantaneous, but arbitrated by some constant time delay τ due to prey handling and digesting [20]. Therefore, we consider the system (3.2) with time delayed term in this section, it can be given by

$$\begin{cases} {}_0D_t^q x(t) = x - x^2 - xk_1(x,y), \\ {}_0D_t^q y(t) = -dy - ey^2 + y(t-\tau)k_2(x(t-\tau), y(t-\tau)), \end{cases} \tag{4.1}$$

where $x(t) > 0, y(t) > 0$, $t \in (-\tau, 0)$ and $q \in (0,1)$. In this part, we consider the local asymptotic stability of equilibrium points for time delayed system (4.1). Correspondingly, the stability of equilibrium points can be analyzed by the following theorem.

Theorem 4.1. *With respect to the stability of equilibrium point of system (4.1), we have that*

(i) the equilibrium point $E_1 = (0,0)$ is unstable;

(ii) the equilibrium point $E_2 = (1,0)$ is locally asymptotically stable, if $d > \frac{f}{1+a_1}$;

(iii) the equilibrium point $E_3 = (x^*, y^*)$ is locally asymptotically stable, if $a_r > 0$, $b_r > 0$, $c_r > 0$, $d_r > 0$ and $b_r - \frac{c_r}{a_r} > 0$, $\frac{a_r^2 d_r - a_r b_r c_r + c_r^2}{c_r - a_r b_r} > 0$;

where $a_r = -2(m_{11} + m_{22})cos(q\pi/2)$, $b_r = m_{11}^2 + 2m_{11}m_{22} + 2m_{11}m_{22}cos(q\pi) + m_{22}^2 - n_{22}^2$, $c_r = 2cos(q\pi/2)n_{22}(m_{11}n_{22} - m_{12}n_{21}) - 2cos(q\pi/2)m_{11}m_{22}(m_{11} + m_{22})$, $d_r = m_{11}^2 m_{22}^2 - (m_{11}n_{22} - m_{12}n_{21})^2$, $m_{11} = \frac{(a_1 + c_1\bar{y})c\bar{x}\bar{y}}{k_4(\bar{x},\bar{y})} - \bar{x}$, $m_{12} = \frac{-c\bar{x}(1+a_1\bar{x})}{k_4(\bar{x},\bar{y})}$, $m_{22} = -d - 2e\bar{y}$, $n_{21} = \frac{f\bar{y}(1+b_1\bar{y})}{k_4(\bar{x},\bar{y})}$ and $n_{22} = \frac{f\bar{x}(1+a_1\bar{x})}{k_4(\bar{x},\bar{y})}$.

Proof. The Jacobian matrix of the linearized system of model (4.1) is

$$J = \begin{bmatrix} 1 - 2\bar{x} - \frac{c\bar{y}(1+b_1\bar{y})}{k_4(\bar{x},\bar{y})} & \frac{-c\bar{x}(1+a_1\bar{x})}{k_4(\bar{x},\bar{y})} \\ \frac{f\bar{y}(1+b_1\bar{y})}{k_4(\bar{x},\bar{y})} e^{-\tau s} & -d - 2e\bar{y} + \frac{f\bar{x}(1+a_1\bar{x})}{k_4(\bar{x},\bar{y})} e^{-\tau s} \end{bmatrix},$$

where $(\bar{x}, \bar{y})$ is the equilibrium point of the system (4.1). The characteristic equation at the equilibrium point E_1 reduces to

$$(s^q - 1)(s^q + d) = 0.$$

Obviously, the characteristic equation has a positive root $s^q = 1$. Therefore, the equilibrium point E_1 is unstable.

For E_2, letting $x(t) = \bar{x} + z(t)$ and $y(t) = \bar{y} + w(t)$ and linearizing the system (4.1), we have

$${}_0D_t^q P(t) = MP(t) + NP(t-\tau), \tag{4.2}$$

where $P(t) = [z(t), w(t)]$, $P(t-\tau) = [z(t-\tau), w(t-\tau)]$ and $q \in (0, 1)$.

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, N = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix},$$

with $m_{11} = \frac{(a_1+c_1\bar{y})c\bar{x}\bar{y}}{k_4(\bar{x},\bar{y})} - \bar{x}$, $m_{12} = \frac{-c\bar{x}(1+a_1\bar{x})}{k_4(\bar{x},\bar{y})}$, $m_{21} = 0$, $m_{22} = -d - 2e\bar{y}$, $n_{11} = 0$, $n_{12} = 0$, $n_{21} = \frac{f\bar{y}(1+b_1\bar{y})}{k_4(\bar{x},\bar{y})}$ and $n_{22} = \frac{f\bar{x}(1+a_1\bar{x})}{k_4(\bar{x},\bar{y})} = -\frac{f}{c}m_{12}$.

With respect to E_2 of the system (4.2), we have

$$M_1 = \begin{bmatrix} -1 & \frac{-c}{1+a_1} \\ 0 & -d \end{bmatrix} \text{ and } N_1 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{f}{1+a_1} \end{bmatrix}.$$

Then, we can obtain

$$\triangle(s) = \begin{bmatrix} s^q + 1 & \frac{c}{1+a_1}, \\ 0 & s^q + d - \frac{f}{1+a_1}e^{-\tau s} \end{bmatrix}.$$

According to Theorem 3.2, when $\tau = 0$ and $d - \frac{f}{1+a_1} < 0$, It is clear that the equilibrium point E_2 is locally asymptotically stable. When $\tau > 0$, the corresponding characteristic equation is given by

$$det(\triangle(s)) = (s^q + 1)(s^q + d - \frac{f}{1+a_1}e^{-s\tau}) = 0.$$

Then, we have $s^q + 1 = 0$ or $s^q + d - \frac{f}{1+a_1}e^{-s\tau} = 0$. Assuming that $s = \omega i$ is a root of the characteristic equation, we can get $s^q = (\omega i)^q = |\omega|^q e^{\frac{q\pi}{2}i} = |\omega|^q(cos(\frac{q\pi}{2}) + isin(\frac{q\pi}{2}))$. If $s^q + 1 = 0$, we have

$$|\omega|^q(\cos(\frac{q\pi}{2}) + i\sin(\frac{q\pi}{2})) + 1 = 0.$$

Then, we can obtain

$$\begin{cases} |\omega|^q \cos(\frac{q\pi}{2}) = -1, \\ \sin(\frac{q\pi}{2}) = 0. \end{cases} \tag{4.3}$$

It is clear that $\frac{q\pi}{2} \in (0, \frac{\pi}{2})$. We have $\sin(\frac{q\pi}{2}) > 0$. Therefore, $s^q + 1 = 0$ does not exist pure imaginary roots.

Similarly, if $s^q + d - \frac{f}{1+a_1}e^{-s\tau} = 0$, we can obtain

$$|\omega|^q(\cos(\frac{q\pi}{2}) + i\sin(\frac{q\pi}{2})) + d - \frac{f}{1+a_1}(\cos(\omega\tau) - i\sin(\omega\tau)) = 0.$$

Then, we can get

$$\begin{cases} |\omega|^q \cos(\frac{q\pi}{2}) + d = \frac{f}{1+a_1}\cos(\omega\tau), \\ |\omega|^q \sin(\frac{q\pi}{2}) = -\frac{f}{1+a_1}\sin(\omega\tau). \end{cases} \tag{4.4}$$

Furthermore, we have

$$|\omega|^{2q} + 2d|\omega|^q \cos(\frac{q\pi}{2}) + d^2 - \frac{f^2}{(1+a_1)^2} = 0.$$

Letting $|\omega|^q = v > 0$, it can be obtained that

$$v^2 + 2dv\cos(\frac{q\pi}{2}) + d^2 - \frac{f^2}{(1+a_1)^2} = 0. \tag{4.5}$$

Assuming that v_1 and v_2 are the roots of (4.5), we have

$$\begin{cases} v_1 + v_2 = -2d\cos(\frac{q\pi}{2}) < 0, \\ v_1 v_2 = (d - \frac{f}{1+a_1})(d + \frac{f}{1+a_1}) > 0. \end{cases} \tag{4.6}$$

There are only negative roots in (4.5) by (4.6). This is in contradiction with the assumption $|\omega|^q = v > 0$. According to Lemma 2.5, E_2 is locally asymptotically stable.

For E_3, similarly, we can obtain the corresponding characteristic matrix

$$\triangle(s) = \begin{bmatrix} s^q - m_{11} & -m_{12} \\ -n_{21}e^{-s\tau} & s^q - m_{22} - n_{22}e^{-s\tau} \end{bmatrix},$$

and characteristic equation

$$det(\triangle(s)) = (s^q - m_{11})(s^q - m_{22} - n_{22}m_{12}e^{-s\tau}) - m_{12}n_{21}e^{-s\tau} = 0. \tag{4.7}$$

Assuming that $s = \omega i$ is a root of the characteristic equation (4.7), we can obtain

$$\begin{cases} [n_{22}|\omega|^q\cos(q\pi/2) - m_{11}n_{22} + m_{12}n_{21}]\cos(\omega\tau) + n_{22}|\omega|^q\sin(q\pi/2)\sin(\omega\tau) \\ = |\omega|^{2q}\cos(q\pi) - |\omega|^q(m_{11} + m_{22})\cos(q\pi/2) + m_{11}m_{22}, \\ n_{22}|\omega|^q\sin(q\pi/2)\cos(\omega\tau) + [m_{11}n_{22} - m_{12}n_{21} - n_{22}|\omega|^q\cos(q\pi/2)]\sin(\omega\tau) \\ = |\omega|^{2q}\sin(q\pi) - |\omega|^q(m_{11} + m_{22})\sin(q\pi/2). \end{cases} \tag{4.8}$$

Furthermore, we have

$$\begin{aligned} &|\omega|^{4q} - 2(m_{11} + m_{22})\cos(q\pi/2)|\omega|^{3q} + (m_{11}^2 + 2m_{11}m_{22} + 2m_{11}m_{22}\cos(q\pi) \\ &+ m_{22}^2 - n_{22}^2)|\omega|^{2q} + [2\cos(q\pi/2)n_{22}(m_{11}n_{22} - m_{12}n_{21}) \\ &- 2\cos(q\pi/2)m_{11}m_{22}(m_{11} + m_{22})]|\omega|^q + m_{11}^2m_{22}^2 - (m_{11}n_{22} - m_{12}n_{21})^2 = 0. \end{aligned}$$

Letting $|\omega|^q = u > 0$, we can obtain

$$u^4 + a_r u^3 + b_r u^2 + c_r u + d_r = 0, \tag{4.9}$$

where $a_r = -2(m_{11} + m_{22})\cos(q\pi/2)$, $b_r = m_{11}^2 + 2m_{11}m_{22} + 2m_{11}m_{22}\cos(q\pi) + m_{22}^2 - n_{22}^2$, $c_r = 2\cos(q\pi/2)n_{22}(m_{11}n_{22} - m_{12}n_{21}) - 2\cos(q\pi/2)m_{11}m_{22}(m_{11} + m_{22})$ and $d_r = m_{11}^2m_{22}^2 - (m_{11}n_{22} - m_{12}n_{21})^2$.

According to Routh-Hurwitz Criterion, this is in contradiction with our hypothetical condition. Therefore, the characteristic equation (4.7) does not exist any pure imaginary roots. Based on Lemma 2.5, the equilibrium point E_3 is locally asymptotically stable. $\square$

5. Numerical simulation

In order to show the analytical stability results obtained in the previous sections graphically, the numerical simulations of systems are conducted respectively.

The system without time delay: we consider the following set of parametric values: $a_1 = 2$, $b_1 = 3.2$, $c = 2.4$, $d = 0.7$, $e = 0.3$, $f = 0.7$, $c_1 = 2$ and $q = 0.7$. Then, the system becomes

$$\begin{cases} {}_0D_t^{0.7}x(t) = x - x^2 - \frac{2.4xy}{1+2x+3.2y+2xy}, \\ {}_0D_t^{0.7}y(t) = -0.7y - 0.3y^2 + \frac{0.7xy}{1+2x+3.2y+2xy}. \end{cases} \tag{5.1}$$

It is easy to find that the parameters of the system (5.1) satisfy the condition $d > \frac{f}{1+a_1}$. We select initial condition $x(0) = 4$, $y(0) = 3$. It can be seen from Figure 2 that the result of the numerical simulation is in line with the conclusion of Theorem 3.2.

Selecting another set of parameters for system (3.2): $a_1 = 1$, $b_1 = 0.5$, $c = 0.1$, $d = 0.1$, $e = 0.3$, $f = 0.6$, $c_1 = 0.1$ and $q = 0.7$, we can find that the corresponding system has a nontrivial equilibrium $E^* \approx (0.9772462, 0.5207189) \in R_+^2$. These parameters obviously satisfy the corresponding conditions of Theorem 3.2. Selecting initial condition $x(0) = 0.6$, $y(0) = 0.3$, according to Figure 3, E^* is a stable equilibrium point and this is in line with the conclusion of Theorem 3.2.

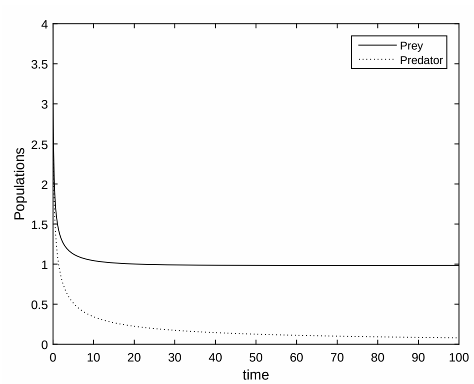

Figure 2. Stability of the equilibrium point $E_1 = (1, 0)$.

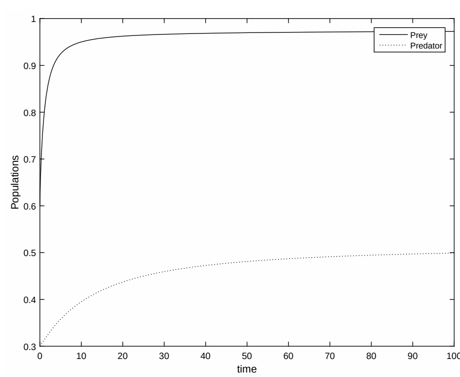

Figure 3. Stability of the equilibrium point $E^* \approx (0.9772462, 0.5207189)$.

Then, we consider another set of parametric values: $a_1 = 0.1$, $b_1 = 0.1$, $c = 0.1$, $d = 0.1$, $e = 0.8$, $f = 0.6$, $c_1 = 0.1$and $q = 0.7$. With these parameters, it is easy to calculate the corresponding nontrivial equilibrium point is $(0.9596775, 0.4798384)$. The condition of Theorem 3.2 is not met. However, according to Figure 4, we can find that $E^* \approx (0.9596775, 0.4798384)$ is still a stable equilibrium point. This shows that the conditions for judging the stability of the equilibrium point in Theorem 3.2 are sufficient and not necessary, which can also be seen from the proof of Theorem 3.2.

The system with time delay: For system (4.1), we select the following set of parameters: $a_1 = 2$, $b_1 = 3.2$, $c = 3$, $d = 0.7$, $e = 0.3$, $f = 1.5$, $c_1 = 2$, $\tau = 0.2$ and

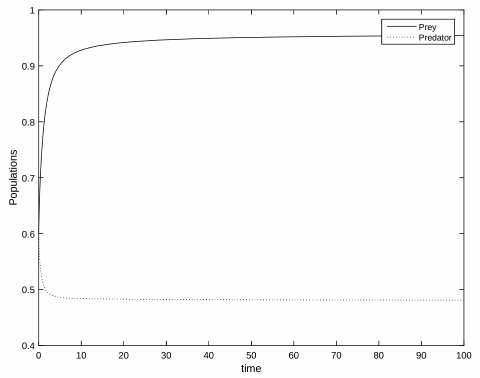

Figure 4. Stability of the equilibrium point $E^* \approx (0.9596775, 0.4798384)$.

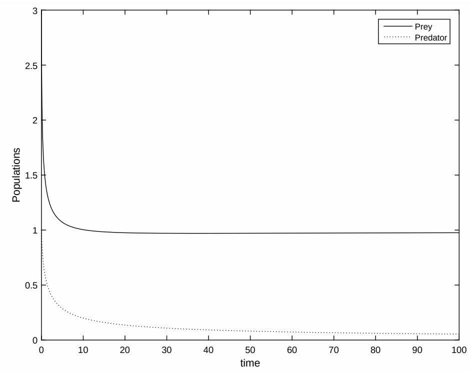

Figure 5. Stability of the equilibrium point $E_2 = (1, 0)$.

$q = 0.7$. It satisfies the condition of Theorem 4.1. Then, we choose initial condition $x_0 = 0.6$, $y_0 = 0.6$. The equilibrium point $E_2 = (1, 0)$ is locally asymptotically stable and the result of numerical simulation is in line with the conclusion of the Theorem 4.1 (Figure 5).

For system (4.1), we set $x_0 = 0.5$, $y_0 = 0.6$. Selecting parameters $a_1 = 0.1$, $b_1 = 0.1$, $c = 0.6$, $d = 0.8$, $e = 0.7$, $f = 1.6$, $c_1 = 0.1$, $\tau = 0.3$ and $q = 0.7$, we can get the nontrivial equilibrium point is $(0.7846384, 0.4135931)$. It is easy to find that this set of parameters satisfies Theorem 4.1. The equilibrium point $E_3 \approx (x^*, y^*)$ is locally asymptotically stable and the result of numerical simulation is in line with the conclusion of the Theorem 4.1 (Figure 6).

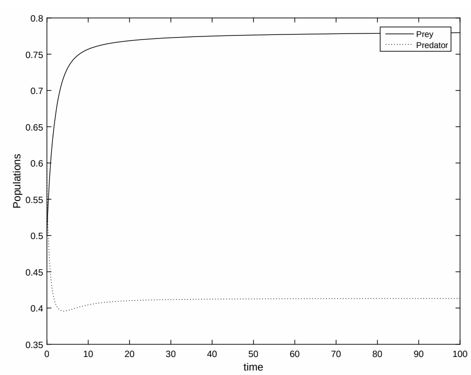

Figure 6. Stability of the equilibrium point $E_3 \approx (0.7846384, 0.4135931)$.

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